PHYSICS 221 FINAL EXAM: SOLUTIONS

Fall, 2008

Problem 1: [15pt]

A helicopter takes off from the ground, and its height above the ground is given by $y = 0.40\, t^3$, where y is in meters and t is in seconds.

(a) [5pt] What is the speed of the helicopter as a function of a time? Give an equation for v as a function of t and a numerical factor. As in the above expression for the height, you don't have to show your units, but should assume v is in $\mathrm{m/s}$ and t is in seconds.

$$v = \frac{dy}{dt} = 0.40\frac{dt^3}{dt} = 0.40\ \times 3t^2 = \boxed{1.2\, t^2.}$$

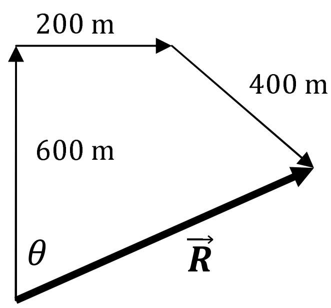
(b) [5pt] The helicopter releases a package at $t = 5.00$ s. At what speed does the package hit the ground, neglecting air resistance? [If you could not answer the previous question, assume the helicopter was moving upward at $20.0\ \mathrm{m/s}$ at the instant when the package was released.]

The package starts moving at the speed of the helicopter, $v_i = 1.2 \times (5.00)^2\ \mathrm{m/s} = 30\ \mathrm{m/s}$. The speed at which the package hits the ground may be obtained using the equation $v_f^2 = v_i^2 + 2gy$, with $y = 0.4 \times (5.00)^3\mathrm{m} = 50\ \mathrm{m}$. Then $v_f^2 = 1880\ \mathrm{m^2/s^2}$, and $v_f = \boxed{43.4\ \mathrm{m/s}.}$

(c) [5pt] If the package's mass is $10.0\ \mathrm{kg}$, how much work did the helicopter's engines have to do to lift the package to the point where it was released? [Assume the package was released with the same velocity that you used in part (b). It is not released at rest.]

The package's energy just before it hit the ground was $E_f = \frac{1}{2}mv_f^2$. The net change in gravitational energy for the round trip is zero, so E_f is the result of just the work done by the helicopter. This implies that $W = \frac{1}{2}mv_f^2 = 5.0\mathrm{kg} \times 1880\ \mathrm{m^2/s^2} = \boxed{9400\ \mathrm{J}.}$ The part of the work used to lift the package to 50 m was $mgh = 4900$ J. The remaining work gives the package its initial kinetic energy, $\frac{1}{2}mv_i^2 = 4500$ J.

Problem 2: [15pt] A boy runs across a field in three straight segments, in a total time of exactly 3 minutes. He first runs $600\ \mathrm{m}$ in due north. He then runs $200\ \mathrm{m}$ due east. He then runs $400\ \mathrm{m}$ south-east (a compass heading of 135^{o} east of north). The times for these segments are 86.0 s, 28.0 s, and 66.0 s, respectively.



(a) [4pt] What was the boy's average speed for the entire race?

The average speed is the total distance traveled divided by the total time,

$$\frac{600\ \mathrm{m} + 200\ \mathrm{m} + 400\ \mathrm{m}}{180\ \mathrm{s}} = \boxed{6.67\frac{\mathrm{m}}{\mathrm{s}}.}$$

Note that this is *not* the average of the speeds for the three sections.

(b) **[7pt]** What are the magnitude and direction of the displacement vector $\vec{\boldsymbol{R}}$ from his starting point to her ending point? Give the direction as a compass heading east of north (θ in the figure).

The total easterly displacement is $x = 200\ \mathrm{m} + (400\ \mathrm{m})\cos 45^{\mathrm{o}} = 483\ \mathrm{m}$ and the total northerly displacement is $y = 600\ \mathrm{m} - (400\ \mathrm{m})\sin 45^{\mathrm{o}} = 317\ \mathrm{m}$. The magnitude and direction are

$$R = \sqrt{x^2 + y^2} = \boxed{578\ \mathrm{m}}$$

$$\theta = \mathrm{atan}\left(\frac{x}{y}\right) = \boxed{56.7^{\mathrm{o}}}$$

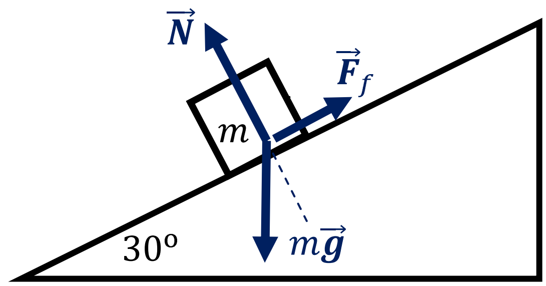
(c) **[4pt]** What are the magnitude and direction of the average velocity vector $\vec{\boldsymbol{v}}_{\mathrm{avg}}$ for the entire three-segment run?

The average velocity vector is the displacement vector divided by the time: $\vec{\boldsymbol{v}}_{\mathrm{avg}} = \vec{\boldsymbol{R}}/\mathrm{t}$.

Therefore, $v_{\mathrm{avg}} = R/t$ and the direction is the same as in part (b):

$$\boxed{v_{\mathrm{avg}} = 3.21\frac{\mathrm{m}}{\mathrm{s}}, \qquad \theta = 56.7^{\mathrm{o}}.}$$

Problem 3: [15pt] A 3.00 kg block starts from rest at the top of a 30^o incline and slides a distance of 2.00 m down the incline in 1.50 s. Do not neglect friction.



(a) [5pt] What is the magnitude of the block's acceleration?

The distance traveled is

$$d = 2.00 \text{ m} = \frac{1}{2}at^2 = \frac{1}{2}a(1.50 \text{ s})^2$$

giving

$$a = \frac{4.00 \text{ m}}{(1.50 \text{ s})^2} = \boxed{1.78 \text{ m/s}^2}$$

(b) [5pt] What is the magnitude of the frictional force between the block and the plane?

Newton's law along the direction of the plane gives

$$ma = mg\sin 30^o - F_f$$

where F_f is the frictional force. Then $F_f = 3.0 \times 3.12 \text{ N} = \boxed{9.36 \text{ N.}}$

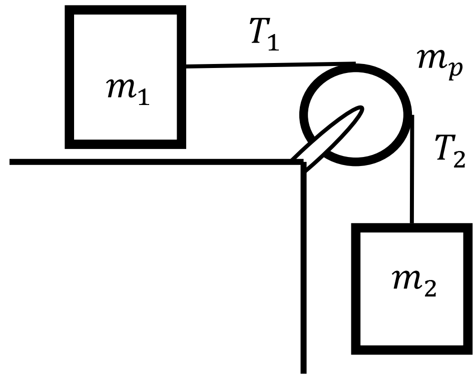
(c) [5pt] What is the coefficient of kinetic friction between the block and the plane?

Balancing forces perpendicular to the plane gives

$$N = mg\cos 30^o = 25.5 \text{ N.}$$

Then the coefficient of friction is $\mu = F_f/N = \boxed{0.367.}$

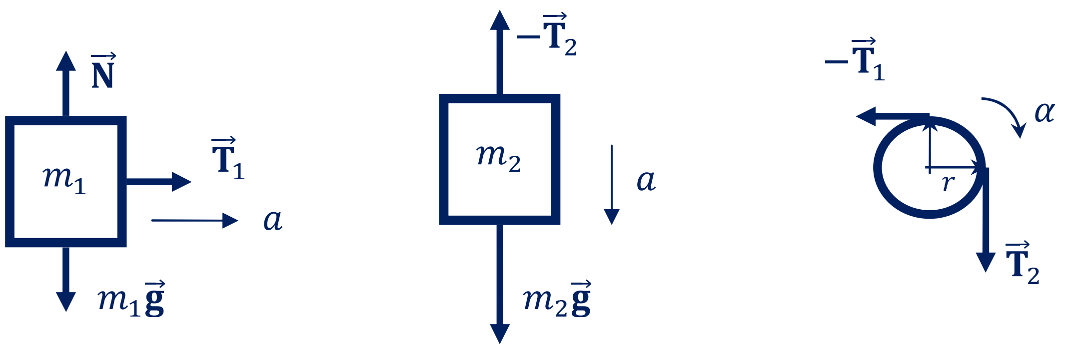
Problem 4: [15pt]



A object of mass $m_1 = 4.50$ kg placed on a horizontal table is connected to a light-weight string that passes over a pulley and then is fastened to a hanging object of mass $m_2 = 6.50$ kg, as shown in the figure. The pulley is a uniform cylinder of mass m_p =1.20 kg and radius $r = 15.0$ cm. Neglect friction.

The moment of inertia of a uniform cylinder *is* $I = ½\, mr^2$.

Find the acceleration of the blocks and the tensions T_1, T_2 in the two parts of the string shown. [If you cannot solve the problem with $m_p \neq 0$, you may solve it with $m_p = 0$ for a maximum credit of 8 points.]



Newton's second law for the two boxes gives $m_1 a = T_1,\ m_2 a = m_2 g - T_2$. (The vertical equation $N = m_1 g$ for the box on the table is not used.)

For the pulley, $\tau = I\alpha = Ia/r$ with $\tau = r(T_2 - T_1)$ and $I = \frac{1}{2}m_p r^2$. This implies that $T_2 - T_1 = \frac{1}{2}m_p a$. Substituting the box results $T_1 = m_1 a,\ T_2 = m_2(a - g)$ into this expression gives $m_1 a + m_2 a - m_2 g = \frac{1}{2}m_p a$. Solving for the acceleration then gives

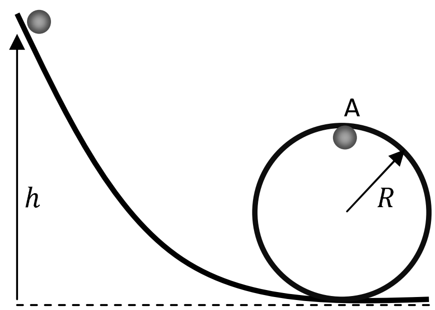
$$a = \frac{m_2 g}{m_1 + m_2 + \frac{1}{2}m_p} = \frac{6.5}{12.2}g = \boxed{5.22\ \mathrm{m/s^2}}$$

Substituting the acceleration into the expressions for the tensions gives

$$T_1 = m_1 a = \boxed{23.5\ \mathrm{N},} \qquad T_2 = m_2(g - a) = \boxed{29.8\ \mathrm{N}.}$$

Neglecting m_p would give $a = 5.79\ \mathrm{m/s^2}$ and $T_1 = T_2 = 26.1$ N.

Problem 5: [12pt] A small bead **slides** without friction around a loop-the-loop track as shown in the figure, starting from rest at height h above the ground, and making a loop of radius R. Neglect the radius of the bead.



(a) [5pt] Find the speed of the bead at point A at the top of the loop in terms of h, R, and g.

Energy conservation implies that $K = \frac{1}{2}mv^2 = mg(h - 2R)$ at point A. Therefore,

$$v = \sqrt{2g(h - 2R)}.$$

(b) [5pt] What is the minimum height h from which the bead can be released to remain in contact with the track at point A, thus making it around the loop without falling off? Express your answer in terms of R and a numerical constant.

The minimum height is that which gives normal force $N = 0$ when the bead reaches point A. This means the bead is in free-fall at point A, acted on only by gravity, so that the centripetal acceleration is $a_c = v^2/R = g$. Combining this with the previous result gives $gR = v^2 = 2g(h - 2R)$, so that

$$h = \frac{5}{2}R.$$

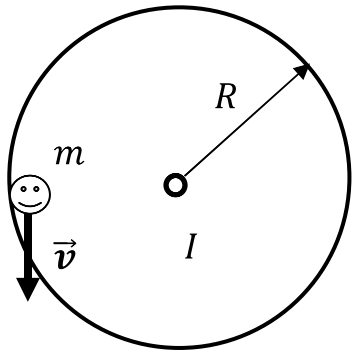
(c) [2pt] If the bead rolls instead of sliding, how would the minimum height to make it around the loop compare to the answer to part (b)?

The new minimum height would be

☒ greater. ☐ less. ☐ the same.

The height would be greater, since the bead would still need the same translational kinetic energy at the top of the loop, but it would also have rotational kinetic energy, which could only be obtained from having more initial potential energy.

Problem 6: [13pt] A woman of mass $m = 66.0$ kg stands at the rim of a horizontal turntable having a moment of inertia $I = 500$ $\text{kg}\cdot\text{m}^2$ and a radius of $R = 2.00$ m. The turntable is initially at rest, and is free to rotate about a frictionless, vertical axle through its center. The woman then starts walking around the rim counterclockwise (as viewed from above the system) at a constant speed $v = 1.50$ m/s relative to the ground.



(a) [3pt] Which of the following are conserved when the woman starts walking? Check the relevant boxes.

☐ Kinetic energy ☐ Linear momentum ☒ Angular momentum

Kinetic energy is not conserved because the woman and platter initially have none, but end up with some. Linear momentum is not conserved because the woman had none to begin with, and then had some. Angular momentum is conserved, because there is no external torque about the axle. (There is an external force on it, but no torque.)

(b) [5pt] At what angular speed does the turntable rotate while the woman is walking?

Conservation of angular momentum implies that $mvR + I\omega = 0$. Therefore, the angular velocity is

$$\omega = -\frac{mvR}{I} = -\frac{66 \times 1.5 \times 2}{500}\,\text{s}^{-1} = -0.396\,\frac{\text{rad}}{\text{s}}$$

and the angular speed is $\boxed{0.396\ \text{rad/s.}}$

(c) [5pt] How much work must the woman do in the process of starting her walk and setting the turntable in motion at this constant speed?

The total energy of the woman and rotating turntable is

$$E = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = 74.25\ \text{J} + 39.20\ \text{J} = 113.5\ \text{J}.$$

The woman must therefore do $\boxed{113.5\ \text{J}}$ of work in the process of starting her walk.

Problem 7: [15pt] A block of mass $m_1 = 1.20\ \mathrm{kg}$ moving at $v_i = 2.5\ \mathrm{m/s}$ has an elastic collision with a block of mass $m_2 = 2.3\ \mathrm{kg}$ initially at rest. After the collision, both blocks are observed to move along the same line as the initial block's motion. (It is a one-dimensional collision.)

(a) [5pt] What was the velocity of the center of mass of the two blocks before and after the collision?

The total momentum is the total mass times the center-of-mass velocity:

$$m_1 v_i = (m_1 + m_2) v_{\mathrm{cm}},$$

giving

$$v_{\mathrm{cm}} = \frac{1.2 \times 2.5}{1.2 + 2.3} \frac{\mathrm{m}}{\mathrm{s}} = \boxed{0.857 \frac{\mathrm{m}}{\mathrm{s}}}.$$

Since momentum is conserved, this is the same before and after the collision.

(b) [10pt] Find the final velocities v_1, v_2 of the two blocks after the collision. Include correct signs, consistent with the convention that the initial velocity of block 1 was positive. [This question is independent of part (a).]

Since the collision is elastic, the relative speed is the same before and after the collision: $v_2 - v_1 = v_i$. Momentum conservation implies that $m_1 v_i = m_1 v_1 + m_2 v_2$. Substituting $v_2 = v_i + v_1$ into this equation gives $m_1 v_i = m_1 v_1 + m_2 v_i + m_2 v_1$.

Therefore,

$$v_1 = \frac{m_1 - m_2}{m_1 + m_2} v_i = \frac{-1.1}{3.5} v_i = \boxed{-0.786 \frac{\mathrm{m}}{\mathrm{s}}},$$

$$v_2 = v_i + v_1 = \boxed{1.714 \frac{\mathrm{m}}{\mathrm{s}}}.$$

Problem 8: [20pt] A 4.80 kg wooden cube measuring 20.0 cm on a side floats in water.

[The density of water is 1000 $\mathrm{kg/m^3}$.]

(a) [5pt] What is the vertical distance from the top of the water to the top of the floating cube?

The weight $mg = \rho_c V_c g$ of the cube must equal the buoyant force, which is the weight of the volume V_w of water displaced by the cube: $B = \rho_w V_w g$. This implies that the ratio of the volume of the cube under water to the entire volume of the cube is

$$\frac{V_w}{V_c} = \frac{\rho_c}{\rho_w} = \frac{m}{\rho_w V_c} = \frac{4.8}{1000 \times (0.20)^3} = 0.600.$$

Therefore, the fraction of the height of the cube above water is 0.400, so that the distance from the water to the top of the cube is 0.400$\times$ $20\mathrm{cm} = \boxed{8.0\ \mathrm{cm.}}$

(b) [5pt] How much force would you have to apply to the top of the cube so that it is just barely submerged, with the top of the cube at the surface of the water?

The force F pushing down on the cube is balanced by the increase in buoyant force when an additional 8.0 cm is under water:

$$F = \Delta B = \rho_w g \Delta V = \rho_w g \times (0.20\ \mathrm{m})^2\ \times 0.080\ \mathrm{m} = \boxed{31.36\ \mathrm{N.}}$$

Alternatively, the additional buoyant force is the weight of water occupying the previously un-sunk 40% of the volume of the cube, which is

$$F = \Delta B = 0.40\left(\frac{mg}{0.600}\right) = \frac{2}{3}mg = 3.2\ \mathrm{kg}\ \times 9.8\frac{\mathrm{N}}{\mathrm{kg}} = 31.36\ \mathrm{N.}$$

(c) [10pt] If you suddenly stop pushing down on the cube, it will bob up and down in the water. What is the period of the resulting harmonic motion? Neglect viscosity of the water.

For an object in simple harmonic motion, the maximum acceleration is $a_{max} = A\omega^2$, where A is the amplitude and ω is the angular frequency. The amplitude will be the maximum displacement from equilibrium, which is 8.0 cm, from part (a). By Newton's second law, the maximum acceleration is also the maximum force divided by the mass, $a_{max} = F/m$, where F is the force holding down the cube in part (b). Then

$$\omega^2 = \frac{F}{mA} = \frac{31.36\ \mathrm{N}}{4.8\ \mathrm{kg} \times 0.080\ \mathrm{m}} = 81.67\ \mathrm{s}^{-2},$$

giving the period $T = 2\pi/\omega = \boxed{0.695\ \mathrm{s.}}$ (The force constant is $F/A\ =\ 395\ \mathrm{N/m}$.)