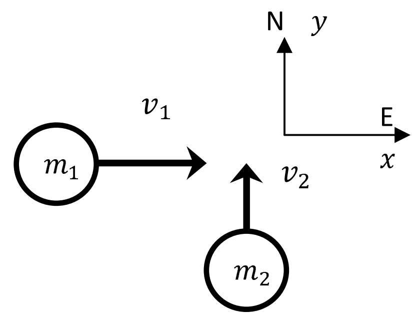
PHYSICS 221 EXAM 3: SOLUTIONS

November 10, 2008

Problem 1: [18pt]



In an American football game, a fullback of mass $m_1 = 95$ kg running east with a speed of $v_1 = 5.8$ m/s is tackled by an opponent of mass $m_2 = 125$ kg running north with a speed of $v_2 = 3.1$ m/s.

[*Note:* This is a two-dimensional problem. It cannot be solved using one-dimensional methods! Vectors are required throughout.]

(a) [6pt] Find the magnitude and direction of the total velocity of the two football players after the tackle. You can give the angle relative to east as the direction.

Momentum conservation in the x direction:

$$(m_1 + m_2)v_f^x = m_1 v_1 \text{ implies } v_f^x = \frac{95}{220} \times 5.8 \text{ m/s} = 2.505 \text{ m/s}.$$

Momentum conservation in the y direction:

$$(m_1 + m_2)v_f^y = m_2 v_2 \text{ implies } v_f^y = \frac{125}{220} \times 3.1 \text{ m/s} = 1.761 \text{ m/s}.$$

$$v_f = \sqrt{\left(v_f^x\right)^2 + \left(v_f^y\right)^2} = \boxed{3.06 \text{ m/s.}} \qquad \theta = \text{atan}\left(\frac{v_f^y}{v_f^x}\right) = \text{atan}\left(\frac{1.761}{2.505}\right) = \boxed{35.1^{\text{o}}.}$$

(b) [6pt] How much kinetic energy was lost by the two players during the collision?

$$K_i = \tfrac{1}{2}m_1 v_1^2 + \tfrac{1}{2}m_2 v_2^2 = 1598 \text{ J} + 601 \text{ J} = 2199 \text{ J}. \qquad K_f = \tfrac{1}{2}(m_1 + m_2)v_f^2 = 1030 \text{ J}.$$

The energy lost during the collision was

$$K_i - K_f = \boxed{1169 \text{ J.}}$$

(c) [6pt] Find the components F_x, F_y of the average force exerted by player 2 on player 1 during their collision, assuming it lasts for a time $\Delta t = 0.50$ s.

The impulse on player 1 is his change in momentum.

In the x direction,

$$F_x t = \Delta p_1^x = m_1(v_f^x - v_1) = (95\text{ kg})\left(-3.295\,\frac{\text{m}}{\text{s}}\right) = -313.0\text{ Ns}, \quad \text{giving}$$

$$F_x = \boxed{-626\text{ N.}}$$

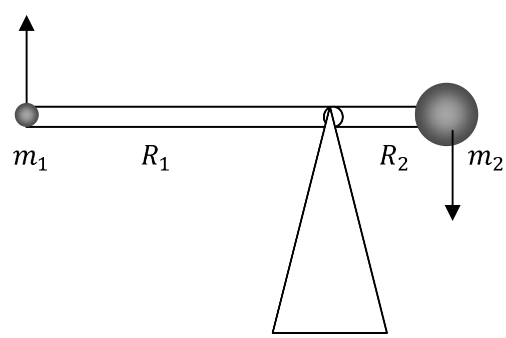
In the y direction, he had no momentum initially, so

$$F_y t = \Delta p_1^y = m_1 v_f^y = (95\text{ kg})\left(1.761\,\frac{\text{m}}{\text{s}}\right) = 167.3\text{ Ns}, \quad \text{giving}$$

$$F_y = \boxed{335\text{ N.}}$$

Alternatively, F_x could be found from the impulse on player 2 and Newton's third law:
$F_x = -\Delta p_2^x = -m_2 v_f^x = -(125\text{ kg})\left(2.505\,\frac{\text{m}}{\text{s}}\right) = -313.1\text{ N}$, giving $F_x = -626\text{ N}$.

Problem 2: [20pt] The trebuchet shown is constructed of a light-weight rod of negligible mass pivoted as shown $R_1 = 7.50$ m and $R_2 = 0.50$ m. A mass $m_2 = 120$ kg is attached to the short end, and allowed to fall, launching mass m_1 =6.0 kg when it gets to its highest point.



(a) [5pt] Find the moment of inertia of the masses and rod, treating the masses as point objects.

$$I = m_1 R_1^2 + m_2 R_2^2 = \boxed{367.5\text{ kg m}^2.}$$

(b) [5pt] Calculate the work done by gravity on the system as it moves from the position shown to the point of launch.

The gravitational work on an object is its weight times its decrease in height. Since m_2 falls a distance R_2 while m_1 is lifted a distance R_1,

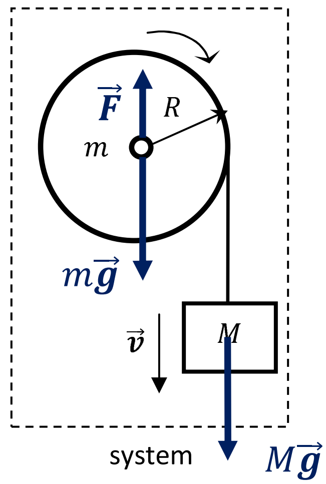
$$W = m_2 g R_2 - m_1 g R_1 = 588\text{ J} - 441\text{ J} = \boxed{147\text{ J.}}$$

(c) [5pt] What is the launch speed of mass m_1?

The launch speed can be found by calculating the rotational kinetic energy, finding that angular speed, and using that to calculate the speed of mass 1. The rotational kinetic energy is $K_{\mathrm{rot}} = \frac{1}{2} I\omega^2 = 147$ J, the work done by gravity. Using the moment of inertia from part (a) gives $\omega = \sqrt{\frac{2\times 147\text{ J}}{367.5\text{ kg m}^2}} = \sqrt{0.800 s^{-2}} = 0.894$ rad/s. The speed of mass 1 is at launch is then $v = R_1\omega = \boxed{6.71\text{ m/s.}}$

(d) [5pt] Find the initial angular acceleration of the trebuchet about its pivot when it is first released from the position shown.

The gravitational torque on the trebuchet in the horizontal position is $\tau = m_2 g R_2 - m_1 g\, R_1 = 147$ Nm. (This appears to be identical to the answer to part (b), but the interpretation is completely different!) The angular acceleration is found using $\tau = I\alpha$and the result from part (a): $\alpha = \frac{147\text{ Nm}}{367.5\text{ kgm}^2} = \boxed{0.400\text{ rad/s}^2.}$



Problem 3: [22pt] A mass M hangs from a spool of string wound around a solid uniform cylinder of mass m and radius R (with moment of inertia $I = ½\, mR^2$). The spool is supported by a rod passing through its center. In this problem, express your results symbolically in terms of quantities labeled in the figure, as needed.

(a) [3pt] Three external forces act on the system consisting of the spool, hanging block, and string: two weights and a force $\vec{F}$ of the rod on the spool. (Tension is an internal force, and not included.) **Draw and label** all three forces in the correct location and direction on the figure to the right. (Put the tail of the vector where it acts.)

(b) [4pt] What is the net torque about the rod due to the external forces?

The weight of the disk and the force of the rod on the spool both act at the center of the disk, and therefore produce no torque about that point. The weight of the box has lever arm R, giving a net torque of

$$\tau = \boxed{MgR.}$$

(c) [5pt] When the block is falling at speed v, what is the total angular momentum of the system about the rod supporting the spool?

The angular momentum of the system is the sum of the angular momentum of the spool, which is $I\omega = \frac{1}{2}mR^2\omega$, and the angular momentum of the falling box, which is its momentum times the distance from its line of motion to the axis, or MvR. Both of these are clockwise angular momenta, so they add with the same sign. (As vectors, they are directed into the page.) Also, $\omega = v/R$, giving total angular momentum

$$L = \frac{1}{2}mRv + MRv = \boxed{\left(M + \frac{m}{2}\right)Rv.}$$

(d) [5pt] Use the answers to parts (b) and (c) together with the fact that $\tau = dL/dt$ to find the acceleration $a = dv/dt$ of the block.

The previous result gives

$$\frac{dL}{dt} = \left(M + \frac{m}{2}\right)R\frac{dv}{dt} = \left(M + \frac{m}{2}\right)Ra.$$

Setting this equal to the torque found in part (b) gives

$$MgR = \left(M + \frac{m}{2}\right) Ra.$$

This gives acceleration

$$\boxed{a = \frac{g}{1 + \frac{m}{2M}}}$$

(e) [5pt] What is the total kinetic energy of the system when the block is falling at speed v?

The total kinetic energy is the sum of the kinetic energy of the falling block and the rotational kinetic energy of the spool:

$$K = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}Mv^2 + \frac{1}{4}mR^2\omega^2 = \boxed{\frac{1}{2}\left(M + \frac{m}{2}\right)v^2}$$

The following problem is extra credit, and can increase your score by a maximum of 3 points. You cannot get more than 60 points on this exam via extra credit, however.

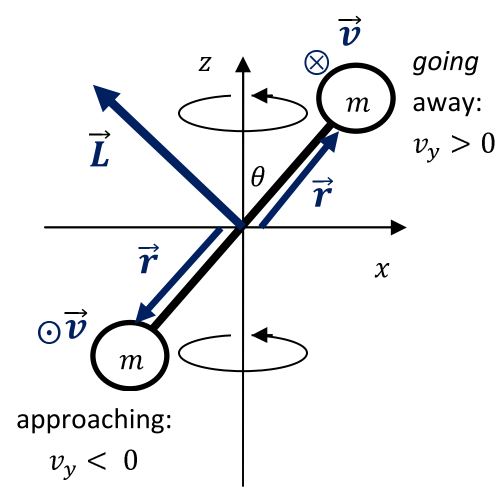
Bonus Question: [3pt] Two equal masses are attached to a tilted rod that rotates counterclockwise at a constant rate about the z axis as shown. At the time shown, both masses are in the plane of the paper. The y axis points into the page.

[2pt] The total angular momentum vector is directed

- **(a)** upward along the z axis.
- **(b)** downward along the z axis.
- **(c)** into the page ($+y$).
- **(d)** out of the page ($-y$).
- **(e)** upward to the left.
- **(f)** downward to the right.
- **(g)** nowhere – it's zero.

$\vec{L} = m\vec{r} \times \vec{v}$.

The right-hand rule gives the same direction for both masses.



[1pt] Is the angular momentum vector conserved?

yes

no

[Circle the correct answers. No explanation is requested.]

The angular momentum vector rotates with the masses, so it can't be conserved. Only the z component is constant.