Extra Credit: Sliding Blocks with Ramp

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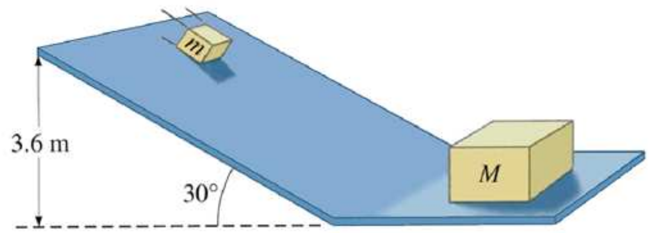

Figure 1: Sliding Blocks

The first question is what is the greatest value of m/M for which the first block will slide down the ramp, bounce up, and come back for a second collision. Let v_0 be the velocity of the small block when it first collides with the large block, which is initially at rest. Assume an elastic collision, and neglect friction. The velocities after the collision are

$$v_1' = v_0 \left(\frac{m - M}{m + M} \right), \qquad v_2' = v_0 \left(\frac{2m}{m + M} \right) \tag{1}$$

for the small and large blocks, respectively. If $m < M$, then v_1 will be negative, meaning the block will head back up the ramp, and then turn around and come back with velocity $-v_1'$, which will be positive. If the relative velocity $v_{12}' = -v_1' - v_2'$ is positive, the small block will catch up to the large one and have a second collision. Adding the velocities in the previous equation gives

$$v_{12}' = v_0 \left(\frac{M - 3m}{m + M} \right). \tag{2}$$

This is positive if $m/M < 1/3$, so the smal block must have less than 1/3 the mass of the large block for a second collision to occur.

The second part of the question asked only if there is a range of masses for which a third collision can occur. This actually just requires a little thought, not a calculation. If the large block were infinitely massive, like a wall, then the first block would bounce back with its original velocity, go back up the ramp, and return with the same velocity. This would continue endlessly, neglecting friction and assuming elastic collisions. Therefore, if m/M is small enough, we should be able to have as many collisions as desired, and in particular, three collisions will occur for some values of m/M.

Limits on Masses for a Third Bounce

Although the problem did not ask for the limits, we can find them by using the elastic collision equations, assuming both blocks are initially moving with velocities $-v_1'$ and v_2'. The equations are at the bottom of page 216 in the textbook. Substituting our notation gives

$$\begin{aligned} v_1'' &= v_1'\left(\frac{M-m}{M+m}\right) + v_2'\left(\frac{2M}{M+m}\right), \\ v_2'' &= v_2'\left(\frac{M-m}{M+m}\right) - v_1'\left(\frac{2m}{M+m}\right) \end{aligned} \tag{3}$$

The relative velocity when the small block returns after the second collision is then

$$v_{12}'' = -v_1'' - v_2'' = -v_1'\left(\frac{M-3m}{M+m}\right) - v_2'\left(\frac{3M-m}{M+m}\right). \tag{4}$$

Substituting the values for v_1' and v_2' from the previous collision gives

$$\begin{aligned} v_{12}'' &= \frac{v_0}{(M+m)^2}\left\{(M-m)(M-3m) - 2m(3M-m)\right\} \\ &= \frac{v_0}{(M+m)^2}(M^2 - 10Mm - 5m^2). \end{aligned} \tag{5}$$

The small block will catch up to the big one if $M^2 - 10Mm - 5m^2 > 0$, or equivalently, if $5(M-m)^2 > 4M^2$. Taking the square root shows that this condition is equivalent to $|M-m| > 2M/\sqrt{5}$, which implies that either of the following conditions must hold:

$$m < \left(1 - \frac{2}{\sqrt{5}}\right)M \quad \text{or} \quad m > \left(1 + \frac{2}{\sqrt{5}}\right)M. \tag{6}$$

The second possibility conflicts with the condition that $m < M/3$ for there to be a second collision. The first condition can be satisfied, and implies that a thirg collision will occur if

$$\frac{m}{M} < 1 - \frac{2}{\sqrt{5}} = 0.106. \tag{7}$$

General Solution

The detailed bound on the masses for a third collision is already more information than the problem asked for, but one may be curious what the condition is for there to be n collisions. This means that the relative velocity $v_{12}^{(n-1)}$ after collision $n-1$ must be positive. The solution is well beyond the scope of the course, but setting it up is not, and you will find the answer at the end.

The starting point is the pair of equations at the bottom of page 216 for a general elastic collision. However, it is convenient to change the sign convention for v_1' so that it is positive, since the small box will be coming back in the case of interest. Making this change gives the elastic collision relations

$$\begin{aligned} v_1' &= v_1\left(\frac{M-m}{M+m}\right) - v_2\left(\frac{2M}{M+m}\right), \\ v_2' &= v_2\left(\frac{M-m}{M+m}\right) + v_1\left(\frac{2m}{M+m}\right) \end{aligned} \tag{8}$$

These equations can then be rewritten in terms of the relative velocity $v_{12} = v_1 - v_2$ and another combination $\hat{v}_{12} = v_1 + v_2$, giving the somewhat simpler relations

$$v_{12}' = 2\rho v_{12} - \hat{v}_{12}, \qquad \hat{v}_{12}' = v_{12}, \tag{9}$$

where $\rho = (M-m)/(M+m)$. Multiple collisions only occur when the small block has less mass than the big one, so $0 < \rho < 1$, in general.

These two equations can be combined for a pair of successive collisions to eliminate $\hat{v}_{12}$, giving

$$v_{12}'' = 2\rho v_{12}' - v_{12} \tag{10}$$

after two collisions. This relation holds for any pair of successive collisions, so we obtain a second order linear difference equation for the relative velocities $v_{12}^{(n)}$ after n collisions:

$$v_{12}^{(n+2)} = 2\rho v_{12}^{(n+1)} - v_{12}^{(n)}. \tag{11}$$

The initial conditions are $v_{12}^{(0)} = v_0$ and from the first part of the problem, $v_{12}^{(1)} = v_0(2\rho - 1)$. The linear difference equation has a unique solution with these initial conditions. If $z_\pm$ are the two roots to the quadratic equation

$$z - 2\rho z + 1 = 0, \tag{12}$$

the general solution has the form

$$v_{12}^{(n)} = a z_+^n + b z_-^n \tag{13}$$

with constants a and b determined by matching this to known results for the cases $n = 0$ and $n = 1$. (This is a lot like solving a second order linear differential equation.) The roots to the quadratic equation are

$$z_\pm = \rho \pm i\sqrt{1-\rho^2} = e^{\pm i\alpha}, \tag{14}$$

with $\alpha = \cos^{-1}(\rho)$. Calculating a and b gives the complete solution

$$v_{12}^{(n)} = \frac{v_0}{\sqrt{1-\rho^2}} \mathrm{Im}\left\{(e^{i\alpha}-1)e^{in\alpha}\right\} = v_0\left\{\cos(n\alpha) - \tan(\alpha/2)\sin(n\alpha)\right\}. \qquad (15)$$

The condition for there to be an $(n+1)$-th collision is that $v_{12}^{(n)} > 0$. This is easiest to solve using the complex-exponential expression for the answer, since $v_{12}^{(n)} = 0$ when $(e^{i\alpha}-1)e^{in\alpha}$ is purely real. This occurs when $e^{(2n+1)i\alpha} = -1$, so that the relative velocity is zero when

$$\alpha = \frac{\pi}{2n+1} + \frac{2\pi k}{2n+1}, \qquad k = 0, \ldots, n. \qquad (16)$$

Not all of these solutions are physically relevant. Since n collisions had to occur already, all of the previous constraints must be satisfied as well. This will be true only for the first solution, $\alpha = \pi/(2n+1)$.

The condition that n collisions will occur is then that $v_{12}^{(n-1)}$ is positive, and the masses must satisfy $\rho > \cos(\pi/(2n-1))$, which means that

$$\frac{M-m}{M+m} > \cos\left(\frac{\pi}{2n-1}\right). \qquad (17)$$

This can be rewritten using a little algebra and trigonometry as

$$\frac{m}{M} < \tan^2\left(\frac{\pi}{4n-2}\right). \qquad (18)$$

Setting $n = 1$ gives $m/M < \tan^2(\pi/2) = \infty$, which just means that the blocks will always collide once, for any values of the masses. Setting $n = 2$ shows that the second collision will occur whenever $m/M < \tan^2(\pi/6) = 1/3$, as expected. Setting $n = 3$ shows that the third collision will occur when $m/M < \tan^2(\pi/10) = 1 - 2/\sqrt{5}$, which is again consistent with the result for that case. When n is large, $\tan(\pi/(4n-2)) \approx \pi/(4n-2) \approx \pi/4n$, so the limiting mass ratio for large n is approximately $m/M < \pi^2/(16n^2)$. When $m/M \to 0$, the blocks can bounce infinitely many times, as expected intuitively.

For given masses, the number of collisions can be found by inverting the constraint on m/M, so that the number of collisions will be the integer part (round down) of

$$n = \frac{\pi}{4\tan^{-1}\sqrt{m/M}} + \frac{1}{2}. \qquad (19)$$

The problem in the exam set $m = 2.20$ kg and $M = 7.00$ kg. In this case, the previous equation gives $n = 2.04$, meaning that there will be two collisions, but not three, for the problem in the exam. This result could also be obtained from the special cases for $n = 2$ and $n = 3$ solved earlier.

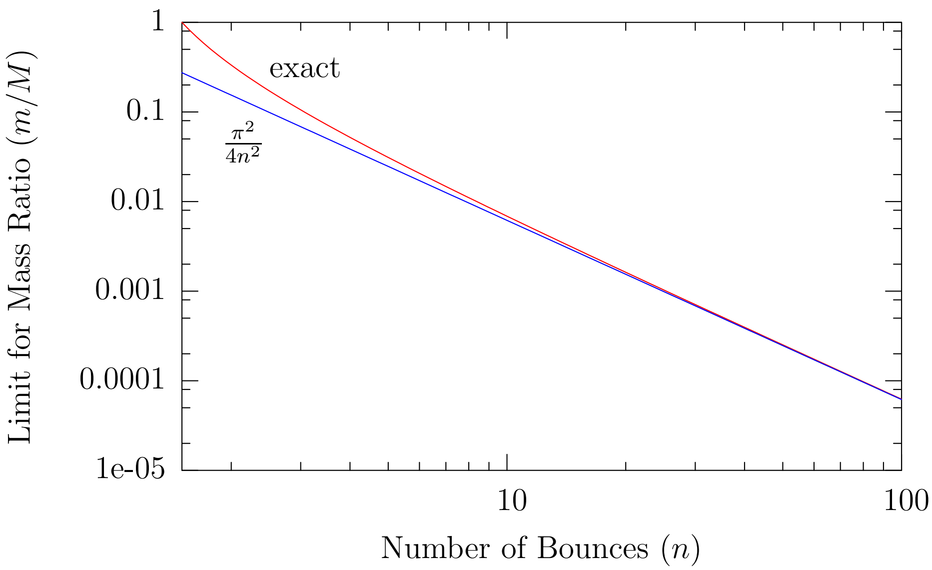

Figure 2: The limit on the mass ratio m/M for the small to large box required for a given number n of bounces. The exact result (18) is compared to the asymptotic expression for large n.

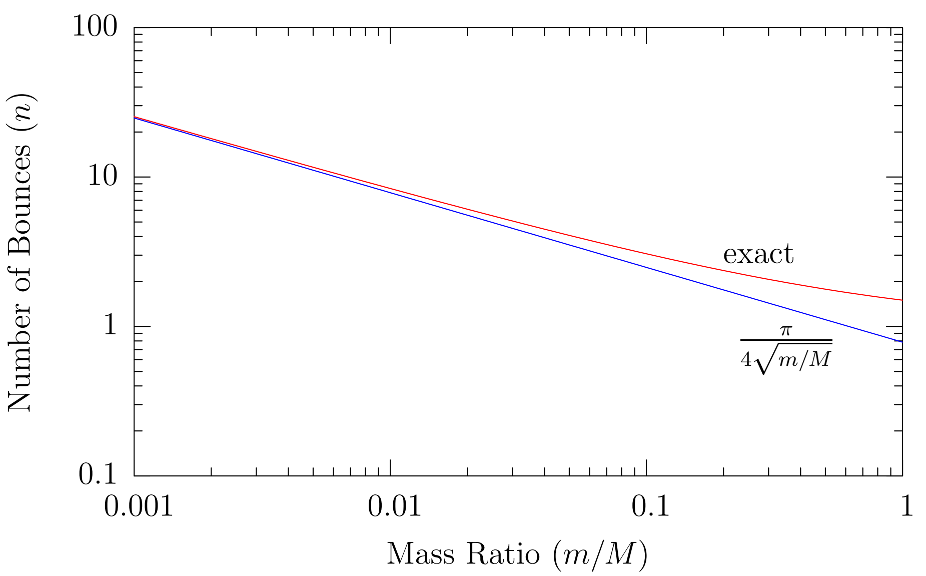

Figure 3: The number of bounces for a given mass ratio m/M of the small to large box. The exact result (19) is compared to the asymptotic expression for large n.