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ADDING HOLES AND CROSSCAPS TO THE SUPERSTRING

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A hole or crosscap in the world sheet of a superstring creates the eigenstate of the quantized boundary conditions. This observation makes the calculation of open string loop divergences almost trivial. We confirm that the O(32) theory is finite. In other theories, the divergences induce BRST anomalies. Cancelling them against the tree-level BRST anomalies of the sigma model gives the loop-corrected equations of motion without ambiguities.

1. Introduction

The complete quantum description of string theory in a general spacetime background is given by the Polyakov path integral sum over nonlinear sigma model amplitudes evaluated on two-dimensional world sheets of arbitrary genus. The resulting expression for string theory amplitudes suffers from two types of divergence: the standard ultraviolet divergences of two-dimensional quantum field theory plus new divergences arising from integrating over the moduli characterizing the conformally inequivalent higher-genus world sheets [1]. The first set of divergences can be eliminated by the usual techniques of renormalization theory, leading to the familiar result that tree-level string physics is summarized by two-dimensional nonlinear sigma models at their renormalization group, or conformal invariance, fixed points. The conformal invariance conditions provide the equations which identify the special spacetimes in which tree-level strings may consistently propagate [2]. Unfortunately, this procedure does nothing to eliminate the second set of divergences which come from integrating over holes, crosscaps or handles added to the tree-level world sheet. Unless these intrinsically string-loop divergences cancel among themselves (which can happen only in very special cases [3,4]), further renormalization is required to make sense of string loop amplitudes.

It has recently been suggested that the extra renormalization needed can be achieved by canceling the two types of divergence against each other [5–7]. This is

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physically plausible since all of the above divergences can be recast as an amplitude for one of the massless states of string theory to disappear into the vacuum [5], and the vacuum stability condition should be that the *sum* of such amplitudes from all sources should vanish. It is far from obvious, from the technical point of view, that such a procedure makes sense, but preliminary investigations show that, at least to first nontrivial order in bosonic string theory, a prescription can be constructed which leads to a consistent set of string-loop-corrected renormalization group beta functions, or generalized conformal invariance conditions, for the spacetime background fields [7]. In this paper we will introduce some technical improvements which allow us to extend our previous work on this subject to the case of superstrings and to explore the important new issue of the rôle of BRST invariance in loop-corrected string theory.

Our technical device applies to the calculation of the effect of adding a hole or crosscap to a closed string world sheet, thus making an open string loop. The hole or crosscap imposes boundary conditions on the world sheet fields. We show that its contribution can be thought of as the projection of the closed string propagator on the eigenstate of those boundary conditions. The string loop divergence is given in turn by the projection of this eigenstate on the zero-mass closed string states. Since the eigenstate is easy to calculate explicitly for bosons, fermions and ghosts, this gives a simple way of extracting complete information about open string loop divergences in all string theories.

A number of advantages flow from this operator approach. We can easily reproduce, without regulator ambiguities, the previous results on loop corrections to bosonic strings as well as the famous result on the cancellation of hole against crosscap divergences in O(32) superstring theory in flat empty spacetime [3,4] (plus the result that these divergences do *not* cancel in a more general background). Having a procedure for dealing with ghost contributions, we can directly compute the loop corrections to the dilaton beta function rather than inferring them by consistency arguments from the metric beta function as was previously necessary [7]. Most importantly, having a convenient formalism for dealing with ghosts allows us to examine the problem of string loop corrections from the point of view of BRST invariance: We find that the tree-level BRST-invariance conditions on the string field [8,9], $Q|\Psi\rangle = 0$, acquire finite corrections from string loops and that the resulting string-loop-corrected BRST-invariance conditions are completely equivalent to the background field equations derived from the divergence cancellation approach. Since BRST invariance is the key to showing how negative metric states decouple from tree-level string amplitudes [8,10,11], this observation should be the kernel of an argument that negative metric states decouple, as they must, from loop-level string amplitudes.

The paper is organized as follows: sect. 2 describes the eigenvector construction of boundary and crosscap operators for the bosonic string. Sect. 3 extends the construction to the superstring, and includes the BRST ghosts. Sect. 4 derives the

BRST anomaly and the loop-corrected background field equations following from its cancellation. Sect. 5 shows the equivalence of these equations with those previously derived from divergence cancellation arguments after a certain field redefinition of the dilaton. Sect. 6 is devoted to a discussion of shortcomings and potential extensions of our results. Sect. 7 summarizes our conclusions. Appendix A verifies that multiplying a boundary bra by a crosscap ket correctly reproduces the Möbius strip amplitude. Appendix B discusses factorization of the partition function for the BRST ghosts and their zero modes. Appendix C summarizes useful facts about the relations betwen various ghost vacua.

2. Boundary eigenstates

We now describe a technical device, which makes open string loop divergence calculations easy. The fundamental open string loop (fig. 1) was first constructed by sewing together the ends of an open string [12]. Later it was discovered [13, 14] that it could be consistently factorized in the closed string channel and regarded as the amplitude for a closed string to disappear into the vacuum. In such a representation, the closed string propagator appears sandwiched between a bra and a ket representing the two ends of the cylinder. The ket is the induced vacuum transition amplitude, and its zero-mass piece is the residue of the divergence. We will show that the bra and ket are just the eigenvectors of the quantized boundary conditions at the two ends of the cylinder. (A similar observation about the three-string vertex was made long ago by Cremmer and Gervais [15].) One normalization constant remains to be determined from open string unitarity. Since the boundary operator must generate the complete amplitude, not just the divergence, there is a vast quantity of finite data to fit and check consistency.

Usually, the boundary of the open string is taken to be $\sigma = \text{constant}$. To ensure that no momentum flows across the boundary, the normal derivative $\partial X^{\mu}/\partial\sigma$ must vanish. Suppose instead that a *closed* string disappears into the vacuum at time τ. The boundary is now $\tau = \text{const}$, so the boundary condition should be

$$\left.\frac{\partial X^{\mu}}{\partial \tau}\right|_{\tau} = 0\,. \tag{2.1}$$

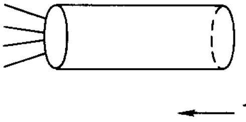

Fig. 1. The loop graph for an open string is the tree graph for a closed string to disappear into the vacuum.

Let us see what this implies after first quantization. The oscillator expansion for a closed string is

$$X(\sigma,\tau)=q-2ip\tau+\sum_{m=1}^{\infty}\frac{1}{\sqrt{m}}\big[a_m^{\dagger}\,\mathrm{e}^{m\tau+im\sigma}+\tilde{a}_m^{\dagger}\,\mathrm{e}^{m\tau-im\sigma}$$
$$+a_m\,\mathrm{e}^{-m\tau-im\sigma}+\tilde{a}_m\,\mathrm{e}^{-m\tau+im\sigma}\big]. \tag{2.2}$$

(The world sheet metric is euclidean, and we have omitted the spacetime index μ.) Our oscillators are related to the α_n of ref. [1] by

$$\alpha_n^{\mu}=-i\sqrt{n}\,a_n^{\mu},\qquad \alpha_{-n}^{\mu}=i\sqrt{n}\,a_n^{\mu\dagger},\qquad n>0. \tag{2.3}$$

We substitute eq. (2.2) into (2.1) and equate coefficients of $\mathrm{e}^{im\sigma}$ to find that the boundary conditions on the modes are

$$p=0,$$
$$\left.\begin{aligned}a_m^{\dagger}\mathrm{e}^{m\tau}&=\tilde{a}_m\mathrm{e}^{-m\tau}\\ \tilde{a}_m^{\dagger}\mathrm{e}^{m\tau}&=a_m\mathrm{e}^{-m\tau}\end{aligned}\right\}\qquad m\geqslant 1. \tag{2.4}$$

The left-moving creation operators must equal the right-moving annihilation operators at time τ, and vice versa.

Let us try to construct an operator which imposes these boundary conditions on the *quantum* fields. The left-moving oscillators see the right-moving ones as c-numbers. Therefore, we have an eigenvalue condition for the annihilation operators. Now the eigenfunction of an annihilation operator is a coherent state

$$a\,\mathrm{e}^{a^{\dagger}f}|0\rangle=f\,\mathrm{e}^{a^{\dagger}f}|0\rangle. \tag{2.5}$$

Therefore the right eigenfunction of eq. (2.4) is

$$|B\rangle\equiv\exp\left(\sum_{m=1}^{\infty}\mathrm{e}^{2m\tau}\,a_m^{\dagger}\tilde{a}_m^{\dagger}\right)|0\rangle. \tag{2.6}$$

Looking at the old factorization papers, we find that their calculations led to the same operator. It gave the amplitudes for states of a closed string to disappear into the vacuum. (See especially Ademollo et al. [14], footnote on p. 228.) Note that $|0\rangle$ in eq. (2.6) represents the closed-string tachyon, not the physical vacuum.

To see the meaning of eq. (2.6), let us put a closed-string tachyon vertex at $\tau+i\sigma\equiv\log z$ in front of a boundary operator at $\tau=\log r$:

$$:\mathrm{e}^{ikX(z)}:\exp\left(\sum_{m=1}^{\infty}r^{2m}a_m^{\dagger}\tilde{a}_m^{\dagger}\right)|0\rangle. \tag{2.7}$$

We split eq. (2.2) into a creation part $X^{(+)}(z)$ and an annihilation part $X^{(-)}(z)$, and contract the annihilation part of the tachyon vertex in eq. (2.7) with the boundary operator $|B\rangle$. By eq. (2.4), we find

$$e^{ikX^{(-)}(z)}|B\rangle = e^{ikX^{(+)}(w)}|B\rangle , \tag{2.8}$$

where

$$w = r^2/\bar{z} . \tag{2.9}$$

If we consider the boundary as a circular mirror of radius $r = e^{\tau}$, then an object at z will produce an image at $r^2/\bar{z}$. The boundary operator therefore reflects vertex operators just as a circular mirror reflects point sources. Neumann functions can be constructed by the method of images, so it is not surprising that sandwiching vertex operators between two boundary eigenstates generates the correct amplitude for the annulus.

The τ-dependence can be factored out of eq. (2.6). It must integrate to give a closed string propagator, i.e.

$$\int_{-\infty}^{0} d\tau e^{-2\tau} \exp\left(\sum_{m=1}^{\infty} e^{2m\tau} a_m^{\dagger} \tilde{a}_m^{\dagger} \right)|0\rangle$$

$$= \int_{-\infty}^{0} d\tau e^{-2\tau} e^{\tau(L_0 + \tilde{L}_0)} \exp\left(\sum_{m=1}^{\infty} a_m^{\dagger} \tilde{a}_m^{\dagger} \right)|0\rangle . \tag{2.10}$$

This determines the integration measure, up to one normalization constant. The closed string propagator pushes the right boundary in fig. 1 along in time, with the limit $\tau \to -\infty$ creating the divergence.

To construct a *left* boundary eigenstate at $\tau = 0$, we just interpret eq. (2.4) as an eigenvalue equation for the *creation* operators, and obtain

$$\langle 0|\exp\left(\sum_{m=1}^{\infty} a_m \tilde{a}_m \right) . \tag{2.11}$$

Taking the scalar product of left and right boundary eigenvectors gives (in D spacetime dimensions)

$$(2.11) \times (2.10) = \int_{-\infty}^{0} d\tau e^{-2\tau} \prod_{m=1}^{\infty} (1 - e^{2m\tau})^{-D} . \tag{2.12}$$

We recognize this as the partition function of the *open* string obtained by cutting the cylinder longitudinally (and making a Jacobi transform) [16].

This construction is easily generalized. In appendix A, we consider inhomogeneous boundary conditions, when open string particles couple to the closed string.

Another generalization is a string interacting with a constant electromagnetic field-strength $F_{\mu\nu}$. The boundary condition in this case becomes

$$\partial X^{\mu}/\partial\tau = -iF^{\mu}_{\nu}\,\partial X^{\nu}/\partial\sigma\,. \tag{2.13}$$

Decomposing into modes by eq. (2.2) and constructing the eigenvector, we easily recover the results of ref. [7]. In this paper, we will take the background $F=0$.

In general, the loop diagrams of an open string include not only the annulus but also the Möbius strip. A Möbius strip can be interpreted as a disk with a crosscap inserted [17]. This means that a circle of radius $r=e^{\tau}$ is cut out of the disk, and its opposite edges identified. The boundary conditions at a crosscap must therefore be

$$\begin{aligned} X(\sigma+\pi,\tau) &= X(\sigma,\tau)\,,\\ \frac{\partial}{\partial\tau}X(\sigma+\pi,\tau) &= -\frac{\partial}{\partial\tau}X(\sigma,\tau)\,. \end{aligned} \tag{2.14}$$

Decomposing these by (2.2), we find

$$\begin{aligned} &p=0\,,\\ &\left.\begin{aligned} a^{\dagger}_{m}\,e^{m\tau} &= (-1)^{m}\tilde{a}_{m}\,e^{-m\tau}\\ \tilde{a}^{\dagger}_{m}\,e^{m\tau} &= (-1)^{m}a_{m}\,e^{-m\tau} \end{aligned}\right\}\qquad m\geqslant 1\,. \end{aligned} \tag{2.15}$$

This differs from eq. (2.4) only by the factor $(-1)^{m}$. The eigenvector $|C\rangle$ for attaching a crosscap of radius $r=e^{\tau}$ is thus found to be

$$|C\rangle = \exp\left(\sum_{m=1}^{\infty}(-1)^{m}\,e^{2m\tau}\,a^{\dagger}_{m}\tilde{a}^{\dagger}_{m}\right)|0\rangle\,. \tag{2.16}$$

The crosscap is just like a boundary of imaginary radius:

$$|C\rangle = i^{L_0+\tilde{L}_0}|B\rangle\,. \tag{2.17}$$

In appendix A, we construct the Möbius strip amplitude this way, and find the correct result.

3. Superstring and superghost boundary operators

We now extend our construction to superstrings and to the ghosts of the covariant formalism [8]. Our reward for care about phase factors will be the ability to discuss BRST invariance questions. The covariant superstring exists in two sectors [1] (Neveu-Schwarz and Ramond), both involving 2D Majorana fermions $\psi(\sigma,\tau)$ with lagrangian

$$\mathscr{L} = i\bar{\psi}\rho^{\alpha}\partial_{\alpha}\psi\,, \tag{3.1}$$

where ρ^{α} are 2×2 Dirac matrices and $\bar{\psi} = \psi\rho^{0}$. The boundary condition at fixed σ is obtained from

$$\delta(\bar{\psi}\rho^{1}\psi) = \delta(\psi_{+}\psi_{+} - \psi_{-}\psi_{-}) = 0$$

giving the usual relation $\psi_{-} = \pm\psi_{+}$ at the edges of the open superstring. However, at fixed τ,

$$\delta(\bar{\psi}\rho^{0}\psi) = \delta(\psi_{+}\psi_{+} + \psi_{-}\psi_{-}) = 0\,,$$

giving

$$\psi_{-} = \pm i\psi_{+}\,. \tag{3.2}$$

The i may seem strange for Majorana fermions, but we will meet several checks that confirm it. The mode expansions of the closed superstring are

$$\psi_{+}(\sigma,\tau) = \sum_{n}\tilde{d}_{n}\mathrm{e}^{-n\tau - in\sigma}\,,$$

$$\psi_{-}(\sigma,\tau) = \sum_{n}d_{n}\mathrm{e}^{-n\tau + in\sigma}\,, \tag{3.3}$$

so at a fixed-τ boundary,

$$d_{n} = \pm i\tilde{d}_{-n}\mathrm{e}^{2n\tau}\,. \tag{3.4}$$

The anticommutators are

$$\{d_{m}, d_{-n}\} = \{\tilde{d}_{m}, \tilde{d}_{-n}\} = \delta_{mn}\,, \tag{3.5}$$

and $n > 0$ gives annihilation operators.

For a single pair of fermion oscillators,

$$\mathrm{e}^{\gamma d^{\dagger}\tilde{d}^{\dagger}}|0\rangle = (1 + \gamma d^{\dagger}\tilde{d}^{\dagger})|0\rangle \tag{3.6}$$

is an eigenfunction of the annihilation operators with

$$d = \gamma\tilde{d}^{\dagger}\,, \qquad \tilde{d} = -\gamma d^{\dagger}\,. \tag{3.7}$$

Therefore, the eigenvector of eq. (3.4) is

$$\exp\Big(\sum_{n>0}(\pm i)\mathrm{e}^{2n\tau}d_{-n}\tilde{d}_{-n}\Big)|0\rangle\,. \tag{3.8}$$

Without the i, there would be no consistent solution. Here n may be an integer or half-integer, depending on the sector of the *closed* string. In the former case, the $n=0$ component of eq. (3.4) must be imposed separately on $|0\rangle$ in eq. (3.8).

The $\pm$ signs in eq. (3.8) correspond to the sectors of the *open* string running transverse to the boundary. The left eigenvector of eq. (3.4) at $\tau=0$ is

$$\langle 0|\exp\Big(\pm i \sum_{n>0} \tilde{d}_n d_n\Big). \tag{3.9}$$

We can construct a cylinder from the scalar product of eqs. (3.9) and (3.8). They represent the two ends. If the $\pm$ signs are the same, then the *open* string flowing *around* the cylinder will have modes d_m with m integer. If the $\pm$ signs are opposite, the open string will have m half-integer. In the NSR formalism, we must combine the two signs to give the GSO projection [18]. In the NS-NS sector of the closed string, n is half-integer and the GSO projection must cancel the spurious tachyon $|0\rangle$. Then eq. (3.8) becomes

$$\sin\Big(\sum_{n=1/2}^{\infty} e^{2n\tau} d_{-n}\tilde{d}_{-n}\Big)|0\rangle. \tag{3.10}$$

Writing $\alpha_n^{\mu} = -i\sqrt{n}\,a_n^{\mu}$, $\alpha_{-n}^{\mu} = i\sqrt{n}\,a_n^{\mu\dagger}$ as in eq. (2.3), the Virasoro generators $L_m^{(\alpha,d)}$ and supergenerators $F_m^{(\alpha,d)}$ of the left-moving modes are

$$L_m^{(\alpha,d)} = \tfrac{1}{2}\sum_n \alpha_{-n}\cdot\alpha_{m+n} + \tfrac{1}{2}\sum_n\big(n+\tfrac{1}{2}m\big)d_{-n}\cdot d_{m+n}, \tag{3.11}$$

$$F_m^{(\alpha,d)} = \sum_n \alpha_{-n}\cdot d_{m+n}. \tag{3.12}$$

Corresponding objects $\tilde{L}_m^{(\alpha,d)}$ and $\tilde{F}_m^{(\alpha,d)}$ are constructed from the right-moving modes $\tilde{\alpha}_n$ and $\tilde{d}_n$. At a fixed-τ boundary we have eq. (3.4), and by eqs. (2.3)–(2.4),

$$\alpha_n = -\tilde{\alpha}_{-n}e^{2n\tau}. \tag{3.13}$$

Thus

$$L_m^{(\alpha,d)} = \tilde{L}_{-m}^{(\alpha,d)}e^{2m\tau},$$
$$F_m^{(\alpha,d)} = \mp i\tilde{F}_{-m}^{(\alpha,d)}e^{2m\tau}. \tag{3.14}$$

In the second term of eq. (3.11), the sign change from $(n+\frac{1}{2}m)$ cancels that from $(\pm i)^2$. This gives one more check of the i in (3.2).

Now we add the BRST ghosts b_n, c_n and the superghosts β_n, γ_n. For the left-moving modes, the BRST charge is [19]

$$\begin{aligned} Q = \sum_n :\left(L_{-n}^{(\alpha,d)} c_n + F_{-n}^{(\alpha,d)} \gamma_n\right): \\ -\tfrac{1}{2} \sum_{m,n} (m-n) : c_{-m} c_{-n} b_{m+n}: \\ + \sum_{m,n} \left(\tfrac{3}{2} n + m\right) : c_{-n} \beta_{-m} \gamma_{m+n}: \\ - \sum_{m,n} : \gamma_{-m} \gamma_{-n} b_{m+n}: - a c_0 . \end{aligned} \tag{3.15}$$

Here, $a = 1$ for the bosonic string, $a = \frac{1}{2}$ for the NS sector of the superstring, and $a = 0$ for the Ramond sector. There is a similar $\tilde{Q}$ constructed from the right-moving modes. We would like the total BRST charge to annihilate the boundary eigenstate, i.e.

$$(Q + \tilde{Q}) |B\rangle = 0 . \tag{3.16}$$

However, $|B\rangle$ turns all annihilation operators into creation operators, rendering the calculation purely classical. So it is necessary and sufficient that $\tilde{Q} = -Q$ classically when the boundary conditions on the modes are inserted. By eq. (3.14) this gives the unique solution at $\tau = 0$:

$$c_n = -\tilde{c}_{-n} , \qquad b_n = +\tilde{b}_{-n} , \tag{3.17}$$

$$\gamma_n = \mp i \tilde{\gamma}_{-n} , \qquad \beta_n = \mp i \tilde{\beta}_{-n} . \tag{3.18}$$

The *same* choice must be made for all upper/lower signs in eqs. (3.4) and (3.18).

The ghosts satisfy

$$\{c_m, b_{-n}\} = [\gamma_m, \beta_{-n}] = \delta_{mn} , \tag{3.19}$$

so the right boundary eigenstate at fixed τ is

$$\exp\left\{ \sum_{n=1}^{\infty} e^{2n\tau} \left[\tilde{c}_{-n} b_{-n} + c_{-n} \tilde{b}_{-n} \right] \mp i \sum_{n>0} e^{2n\tau} \left[\tilde{\gamma}_{-n} \beta_{-n} - \gamma_{-n} \tilde{\beta}_{-n} \right] \right\} (c_0 + \tilde{c}_0) |\downarrow\downarrow\rangle . \tag{3.20}$$

The $n = 0$ part of eq. (3.17) has been imposed separately, starting from the Siegel vacuum $|\downarrow\downarrow\rangle$ of the closed string, which satisfies

$$b_0|\downarrow\downarrow\rangle \equiv \tilde{b}_0|\downarrow\downarrow\rangle \equiv 0\,. \tag{3.21}$$

(See appendix C for a summary of ghost vacuum properties.) In the RR sector of the closed superstring, the superghosts also have zero-modes. These lead to notorious problems which we hope to discuss elsewhere. The upper/lower sign in eq. (3.20) must agree with that in eq. (3.8).

The left eigenvector of eqs. (3.17)–(3.18) is

$$\langle\uparrow\uparrow|\left(b_0-\tilde{b}_0\right)\exp\left\{\sum_{n=1}^{\infty}\left[\tilde{b}_n c_n + b_n\tilde{c}_n\right] \mp i\sum_{n>0}\left[\tilde{\beta}_n\gamma_n - \beta_n\tilde{\gamma}_n\right]\right\}. \tag{3.22}$$

The upper/lower sign in eq. (3.22) must agree with that in eq. (3.9).

Now we consider crosscaps. It is difficult to develop any geometric intuition for what happens when a superghost meets a crosscap. In the bosonic case, eq. (2.15), we found that the crosscap differed from the boundary only by factors of $(-1)^n$ in the nth modes. This leads to a factor of $(-r^2)^n$ in the partition and Neumann functions (A.7). The same change $r^2 \to -r^2$ turns the annulus functions into those of a Möbius strip for the superstring [4]. Furthermore, $[Q, L_0] = 0$, so eq. (2.17) will be BRST invariant after including the ghosts. Except for an overall normalization constant, the crosscap operator must therefore be obtained by inserting $(-1)^n$ in every term of the exponent of the boundary operator.

Now we discuss the normalization of the eigenvectors. This has to be done by comparison with the open-string loop. If N types of "quarks" can run around the right end of fig. 1, we clearly get a factor of N. Coupling the closed string tachyon to an open string just inserts 1 into the amplitude [20]. However, in a φ^3 field theory, a graph with M external lines and L loops has $M+2(L-1)$ vertices. Adding a hole to a graph therefore multiplies it by λ^2, where λ is the open string coupling constant. So a reasonable normalization of the boundary operator is to multiply by $\lambda^2 N$ in a theory with $O(N)$, $U(N)$, or $Sp(N)$ Chan-Paton factors.

The normalization of the crosscap relative to the boundary is of utmost importance, since it generates the anomaly cancellations. The open string Möbius strip has a twist factor [17] $\eta = 1$ for $Sp(N)$, $\eta = 0$ for $U(N)$, and $\eta = -1$ for $O(N)$. There are also various powers of 2. The disc with a crosscap looks very similar to the annulus (fig. 2). However, if we cut them apart to form a rectangle, the latter will be twice as long. This follows from computing the lengths of the boundaries.

For a bosonic string, careful examination of the Jacobi transforms show that we get a factor

$$2^{-M-1}\times 2^{D}\times 2^{(2-D)/2}\times 2^{M} = 2^{D/2} \tag{3.23}$$

in the Möbius strip relative to the annulus in an amplitude with M external lines on

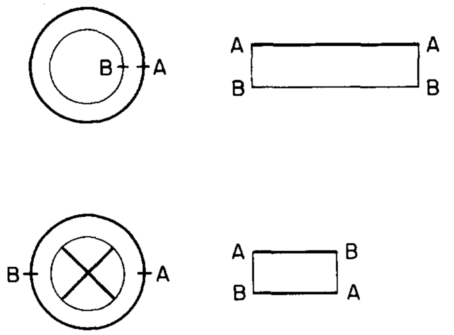

Fig. 2. The disc with a hole has the same circumference as the disc with a crosscap, but the rectangle from which it is sewn must be twice as long.

the outer boundary. Here, D is (as usual) the spacetime dimension. The successive factors on the left-hand side of eq. (3.23) come from the integration measure, the loop momentum integral, the partition functions, and the Neumann functions. Cancellation of M in the product is a crucial consistency check. For the superstring the calculation is different [3] but several authors have found the same result (3.23). If the $O(N)$ open superstring is consistent at all, the normalization factor for the crosscap operator must therefore be $-32\lambda^2$. In the NS-NS sector, this multiplies a sine like eq. (3.10). The phase is determined by the zero-mass state, using eq. (2.17).

4. Loop-corrected field equations

The complete amplitude for annulus and Möbius strip is

$$\langle M|\left[L_0+\tilde{L}_0-2a\right]^{-1}\{|B\rangle+|C\rangle\}, \tag{4.1}$$

Here $\langle M|$ represents the outer boundary with M open string particles attached, as in appendix A. The closed string propagator comes from eq. (2.9) and includes the BRST ghosts. (See after eq. (3.15) for a.) $|B\rangle$ and $|C\rangle$ are the boundary and crosscap operators (evaluated at $\tau=0$). They give the transition amplitudes for various closed string states to disappear into the vacuum. Unless the zero mass component $|B\rangle_0+|C\rangle_0$ cancels, the propagator in eq. (4.1), and the one-loop amplitude as a whole, will diverge.

We have seen that, for the O(32) string, the zero-mass component of the sum of hole and crosscap boundary operators vanishes and the one-loop amplitude is finite. Concern has sometimes been expressed [21] that this divergence cancellation is regulator-dependent and therefore ambiguous. Since the boundary operators are perfectly finite and the relevant physical question from our point of view is whether the net boundary operator has any projection on massless states, it seems to us that there is no room for such a regulator ambiguity. On the other hand, the divergence cancellation, even for O(32) strings, does depend on the background being that of flat empty spacetime. In the presence of a background gauge field, the hole boundary conditions are modified (see refs. [7] and [22] for details) while the crosscap boundary conditions are not (the crosscap does not correspond to an edge of the world sheet and no external field can be attached to it). The cancellation of the vacuum instability is therefore upset under nonvacuum conditions and we will, even for O(32), be faced with the problem of renormalizing string loop divergences.

Specific models in which divergences cancel between string trees and loops have recently been studied [6, 7]. From our current perspective, this is possible because the ultraviolet divergences of the world sheet sigma model can be rewritten as infrared divergences or vacuum instabilities of the string [5]. The rigorous proof is long, but the following heuristic argument may be helpful. We consider an almost free (i.e. flat space) sigma model and expand all amplitudes in the sigma model interaction $\Psi(z)$ (a typical interaction term $\delta g_{\mu\nu}(X)\,\partial_z X^\mu \partial_{\bar{z}} X^\nu$ defines a small perturbation of the background metric away from flat space). To first order there will only be a single insertion, which we can fix at $z=0$ by translation invariance. The correction to the amplitude for a disc will then be $\langle M|\Psi(0)|0\rangle$. This is the tree amplitude for coupling M open strings to one closed string in the state $|\Psi\rangle = \Psi(0)|0\rangle$. We rewrite the amplitude in a formally identical manner as

$$\langle M|\Psi\rangle = -\langle M|\left[L_0+\tilde{L}_0-2a\right]^{-1}|\beta\rangle, \tag{4.2}$$

where for an off-shell massless state,

$$|\beta\rangle = -\left[L_0+\tilde{L}_0-2a\right]|\Psi\rangle = -\tfrac{1}{2}p^2|\Psi\rangle. \tag{4.3}$$

Now, as will become apparent in eq. (4.19) below, $|\beta\rangle$ is a sum of massless string states whose coefficients are essentially the ordinary beta functions of the sigma model defined by $|\Psi\rangle$. If the beta functions vanish, the massless particle poles in eq. (4.2) are eliminated, consistent with the notion that the vanishing sigma model beta function condition is a vacuum stability condition. In the presence of loop divergences, vacuum stability can be maintained by setting

$$-|\beta\rangle + |B\rangle_0 + |C\rangle_0 = 0 \tag{4.4}$$

so as to cancel the pole in eq. (4.2) against that in eq. (4.1). In this manner, the tree level field equation of the string acquires a set of loop corrections which turn out to be identical to the loop-corrected beta functions we have discussed in a previous paper.

At least in tree approximation, the conformal invariance, or vanishing beta function, conditions are equivalent to BRST invariance. For that reason, in this paper we would like to show that loop-corrected string equations can be derived from BRST invariance just as easily as from divergence cancellation. The tree-level string field equations can be elegantly derived from [9]

$$Q_{\mathrm{BRST}}|\Psi\rangle = 0. \tag{4.5}$$

Mansfield [23] (also Martinec) pointed out that string loop divergences would invalidate the BRST Ward identities. Although we do not use his methods directly, we can use our boundary operators to exhibit explicitly the BRST anomaly and show that it can be cancelled by the trees to give the loop-corrected equations of motion.

The boundary and the crosscap operators annihilate $Q + \tilde{Q}$ by construction. Also, $[Q, L_0] = 0$. The anomaly arises because the cylinder needs ghost zero mode insertions. As discussed in appendices B and C, eqs. (3.20) and (3.22) are orthogonal, i.e.

$$\langle\uparrow\uparrow|(b_0 - \tilde{b}_0) \times (c_0 + \tilde{c}_0)|\downarrow\downarrow\rangle = 0. \tag{4.6}$$

This is a familiar feature of the ghost vacua [24]. The cylinder has one Teichmüller parameter (τ in eq. (2.10)), and one conformal symmetry (rotation). So we must insert one b and one c to get a nonzero matrix element. The precise form is not too important, because all $n \neq 0$ terms in the insertion will vanish by eq. (4.6). An attractive convention [25] is to insert $c(z)$ at each vertex and $\oint \mathrm{d}z\, b(z)$ around each propagator. The two will cancel except for c's at the fixed Koba-Nielsen points which gauge-fix the conformal symmetry, and b's for the propagators corresponding to Teichmüller parameters. The fixed Koba-Nielsen point $z_1 = 1$ of the annulus is on the outer boundary. By eq. (3.17), $\tilde{c}(1) = -c(1)$, so the c insertion is

$$\tilde{c}(1) - c(1) \cong \tilde{c}_0 - c_0, \tag{4.7}$$

up to terms which vanish by eq. (4.6). The b insertion is taken around the cylinder. Again up to terms which vanish by eq. (4.6), the insertion is

$$-\int \mathrm{d}\sigma\, b_{\tau\tau}(\sigma, \tau) \cong -\left(b_0 + \tilde{b}_0\right). \tag{4.8}$$

(At present there is no agreed sign convention for ghost insertions. We choose them to get the right final sign.) Now eq. (4.7) is part of the outer boundary $\langle M|$ in eq. (4.1) and matches a similar insertion on the disc. However, eq. (4.8) is part of the propagator in eq. (4.1) and is peculiar to loop graphs.

This has a drastic effect on BRST invariance. We have

$$\left\{Q + \tilde{Q}, b_0 + \tilde{b}_0\right\} = L_0 + \tilde{L}_0 - 2a, \tag{4.9}$$

where the right-hand side also includes the ghost hamiltonian. Thus

$$(Q+\tilde{Q})(-b_0-\tilde{b}_0)\int_{-\infty}^{0} \mathrm{d}\tau\, \mathrm{e}^{\tau(L_0+\tilde{L}_0-2a)}\{|B\rangle+|C\rangle\}$$

$$=-\int_{-\infty}^{0} \mathrm{d}\tau \frac{\mathrm{d}}{\mathrm{d}\tau} \mathrm{e}^{\tau(L_0+\tilde{L}_0-2a)}\{|B\rangle+|C\rangle\}. \tag{4.10}$$

Now as $\tau \to 0$, the cylinder (fig. 1) contracts to a circular wire, whereas for $\tau \to -\infty$ it becomes infinitely long. For a thin wire, we cannot consistently separate the two boundaries, or usefully expand in closed-string oscillators. Instead, we must undo the Jacobi transformation (eq. (B.6)) and expand in w to get the unitarity cut of the open string loop [26]. The divergence and vacuum instability are known to arise solely from the $\tau \to -\infty$ limit. For the purposes of the present paper, we can therefore replace eq. (4.10) by

$$\lim_{\tau \to -\infty} \mathrm{e}^{\tau(L_0+\tilde{L}_0-2a)}\{|B\rangle+|C\rangle\}. \tag{4.11}$$

Now $p=0$ by eq. (2.4), so the exponent is the (mass)^2 operator. The limit $\tau \to -\infty$ will thus cancel all states of positive (mass)^2, give a finite result $|B\rangle_0+|C\rangle_0$ for zero mass states, and a divergence for tachyons. Thus in the superstring theory we have effectively

$$(Q+\tilde{Q})(-b_0-\tilde{b}_0)\left[L_0+\tilde{L}_0-2a\right]^{-1}\{|B\rangle+|C\rangle\}=|B\rangle_0+|C\rangle_0. \tag{4.12}$$

So the BRST anomaly is the residue of the IR divergence.

In sects. 2 and 3, we constructed $|B\rangle$ and $|C\rangle$ explicitly. Their zero-mass parts can be extracted by standard mode-counting. For the O(N) bosonic string,

$$|B\rangle_0+|C\rangle_0=\lambda^2(N-2^{13})\{-\alpha^{\mu}_{-1}\tilde{\alpha}^{\mu}_{-1}+\tilde{c}_{-1}b_{-1}+c_{-1}\tilde{b}_{-1}\}(c_0+\tilde{c}_0)|\downarrow\downarrow\rangle. \tag{4.13}$$

(The 2^{13} agrees with Polyakov-type calculations [27]. The minus sign in the α term comes from eq. (2.3).) The bosonic oscillators in eq. (4.13) correspond to the spacetime metric tensor wave function. The metric alone does not have enough degrees of freedom for a Brans-Dicke theory. To describe the dilaton, we must add either a coupling to the world sheet curvature [28] or to the BRST ghosts [9]. Either method is internally consistent, but their equivalence is nontrivial. The ghost terms in eq. (4.13) agree with Siegel and Zwiebach's dilaton wave function.

For the O(N) superstring, we consider only the NS-NS sector, where by eqs. (3.10) and (3.20),

$$|B\rangle_0+|C\rangle_0$$

$$=\lambda^2(N-32)\{d^{\mu}_{-1/2}\tilde{d}^{\mu}_{-1/2}-\tilde{\gamma}_{-1/2}\beta_{-1/2}+\gamma_{-1/2}\tilde{\beta}_{-1/2}\}(c_0+\tilde{c}_0)|\downarrow\downarrow\rangle. \tag{4.14}$$

The dilaton is now described by a coupling to the superghosts.

Next we construct the tree amplitudes which will cancel the associated divergences and/or BRST anomalies. We consider a bosonic string wave function

$$|\Psi\rangle = \left\{T(q) + h_{\mu\nu}(q)\tilde{\alpha}^{\mu}_{-1}\alpha^{\nu}_{-1} + \Phi(q)\left[\tilde{c}_{-1}b_{-1} + c_{-1}\tilde{b}_{-1}\right]\right\} c(0)\tilde{c}(0)\left|0[SL_2]\right\rangle, \tag{4.15}$$

adopting the interpretation that $|\Psi\rangle$ describes the shift of the original sigma model away from flat empty spacetime. Here q^{μ} is the average position of the closed string in spacetime. $T(q)$ is the tachyon wave function, while $h_{\mu\nu}(q)$ and $\Phi(q)$ correspond to the graviton (plus antisymmetric tensor) and dilaton. The ghost vacuum is that appropriate to the tree [24], but the Koba-Nielsen fixed point $z = \bar{z} = 0$ gives

$$c(0)\tilde{c}(0)\left|0[SL_2]\right\rangle = |\downarrow\downarrow\rangle, \tag{4.16}$$

by eq. (C.14).

Following Siegel and Zwiebach [9], we generate field equations by acting on (4.15) with $Q + \tilde{Q}$. The relevant terms of eq. (3.15) are

$$Q = c_0\left(\tfrac{1}{2}p^2 + \mathcal{N}\right) + c_{-1}(p\cdot\alpha_1) + (p\cdot\alpha_{-1})c_1 + \cdots, \tag{4.17}$$

where $\mathcal{N}$ is the mass operator

$$\mathcal{N} = \alpha_{-1}\cdot\alpha_1 + c_{-1}b_1 + b_{-1}c_1 - 1 + \cdots, \tag{4.18}$$

and $\alpha_0 = p$. Thus

$$\begin{aligned}(Q + \tilde{Q})|\Psi\rangle = {} & \left\{\left(\tfrac{1}{2}p^2 - 2\right)T(q) + \tfrac{1}{2}p^2 h_{\mu\nu}(q)\tilde{\alpha}^{\mu}_{-1}\alpha^{\nu}_{-1} + \tfrac{1}{2}p^2\Phi(q)\left[\tilde{c}_{-1}b_{-1} + c_{-1}\tilde{b}_{-1}\right]\right\} \\ & \times(c_0 + \tilde{c}_0)|\downarrow\downarrow\rangle + \left\{\left[p^{\mu}h_{\mu\nu}(q) - p_{\nu}\Phi(q)\right]\alpha^{\nu}_{-1}\tilde{c}_{-1}\right. \\ & \left. + \left[p^{\nu}h_{\mu\nu}(q) - p_{\mu}\Phi(q)\right]\tilde{\alpha}^{\mu}_{-1}c_{-1}\right\}|\downarrow\downarrow\rangle. \end{aligned} \tag{4.19}$$

Requiring the coefficients of each state to vanish gives free wave equations for the tachyon, graviton, and dilaton, plus

$$\partial^{\mu}h_{\mu\nu} = \partial^{\mu}h_{\nu\mu} = \partial_{\nu}\Phi, \tag{4.20}$$

which gives the gauge conditions for the graviton $h_{(\mu\nu)}$ and antisymmetric tensor $h_{[\mu\nu]}$. These coincide with first order calculations of $\beta = 0$ in the sigma model [2]. (To get the higher order tree corrections, we could use the BRST operator constructed from the perturbed energy-momentum tensor of the sigma model [29].)

For the NS-NS sector of the closed superstring, we take

$$|\Psi\rangle = \left\{ h_{\mu\nu}(q)\tilde{d}^{\mu}_{-1/2}d^{\nu}_{-1/2} + \Phi(q)\left[\gamma_{-1/2}\tilde{\beta}_{-1/2} - \tilde{\gamma}_{-1/2}\beta_{-1/2}\right]\right\}|\downarrow\downarrow\rangle. \quad (4.21)$$

This has zero mass, so the relevant terms of eq. (3.15) are

$$Q = \tfrac{1}{2}p^2 c_0 + p_\mu\left[d^{\mu}_{-1/2}\gamma_{1/2} + d^{\mu}_{1/2}\gamma_{-1/2}\right] + \cdots, \quad (4.22)$$

giving

$$\begin{aligned}(Q+\tilde{Q})|\Psi\rangle = &\left\{\tfrac{1}{2}p^2 h_{\mu\nu}(q)\tilde{d}^{\mu}_{-1/2}d^{\nu}_{-1/2}\right.\\ &\left.+\tfrac{1}{2}p^2\Phi(q)\left[\gamma_{-1/2}\tilde{\beta}_{-1/2} - \tilde{\gamma}_{-1/2}\beta_{-1/2}\right]\right\}(c_0+\tilde{c}_0)|\downarrow\downarrow\rangle\\ &+i\left\{\left[\partial^{\nu}h_{\mu\nu}(q) - \partial_\mu\Phi(q)\right]\tilde{d}^{\mu}_{-1/2}\gamma_{-1/2}\right.\\ &\left.-\left[\partial^{\mu}h_{\mu\nu}(q) - \partial_\nu\Phi(q)\right]d^{\nu}_{-1/2}\tilde{\gamma}_{-1/2}\right\}|\downarrow\downarrow\rangle. \end{aligned} \quad (4.23)$$

Except for the absence of the tachyon, this corresponds closely to eq. (4.19), and should transform into it under the picture-changing operation [30]

$$|\Psi\rangle \to F_{-1/2}\tilde{F}_{-1/2}|\Psi\rangle, \quad (4.24)$$

where F_m is eq. (3.12) plus a superghost piece.

We can now obviously cancel eq. (4.13) against (4.19), and eq. (4.14) against (4.23). This gives the loop corrections to the equations of motion

$$\begin{aligned}\tfrac{1}{2}\nabla^2 h_{\mu\nu} &= -\lambda^2(N - 2^{D/2})\eta_{\mu\nu},\\ \tfrac{1}{2}\nabla^2\Phi &= \lambda^2(N - 2^{D/2}).\end{aligned} \quad (4.25)$$

The left-hand side is first order because we used the flat-space value of $Q+\tilde{Q}$ in eqs. (4.19) and (4.23). The right-hand side is zeroth order, because we only calculated the annulus divergence in flat space. Also, we must not assume that the dilaton Φ coupled to ghosts is the same as the dilaton ϕ coupled to curvature. Even with these limitations, it is not obvious that eqs. (4.25) are consistent. We will devote the next section to showing that they are.

5. Effective lagrangian

We now wish to compare the loop-corrected BRST equations for background fields with the similar equations which follow from the divergence cancellation method. In ref. [7] we used that method to derive the following string-loop-corrected beta functions for a bosonic string in background metric and dilaton fields:

$$R_{\mu\nu} - \nabla_\mu \nabla_\nu \phi = \tfrac{1}{4}\kappa\, e^{-\phi/2} g_{\mu\nu}\,,$$

$$-R + 2\nabla^2\phi + (\nabla\phi)^2 = \tfrac{1}{2}\kappa\, e^{-\phi/2}\,. \tag{5.1}$$

On the left-hand sides of these expressions are the standard sigma model beta functions while on the right-hand sides are the corrections arising from the effect of one open string loop. (Here, κ is a loop normalization constant which depends on the details of the theory.) A particularly important feature of these equations, and one which guarantees their internal consistency, is that they can be derived from the simple spacetime effective action

$$S = \int d^D X \sqrt{g}\left\{ e^{\phi}\left[R + (\nabla\phi)^2\right] + \kappa\, e^{\phi/2}\right\}. \tag{5.2}$$

In this action, and elsewhere, e^{ϕ} plays the role of a string loop expansion parameter and one can use the relative power of this quantity to identify the relative loop order of different terms [31].

Superficially, eq. (5.1) does not look much like the BRST equations, eqs. (4.25) and (4.20). There are several reasons for this, which we will deal with in turn. To start with, the BRST equations have been obtained in weak field approximation. We should therefore set $g_{\mu\nu} = \bar{\eta}_{\mu\nu} + h_{\mu\nu}$ and expand eq. (5.1) to the lowest relevant order in $h_{\mu\nu}$ and $\phi - \bar{\phi}$, where $\bar{\phi}$ is the *constant* dilaton vev which determines the string loop coupling. We expand the tree part (l.h.s.) of eq. (5.1) to first order in these fields, and the loop part (r.h.s.) to zeroth order. At the same time, it is necessary to realize that the BRST equations are in gauge-fixed form (one of them, eq. (4.20), is obviously a spacetime general coordinate gauge condition) so that a gauge choice must be made before any comparison. In fact, along with the on-shell divergences which lead to the gauge-invariant beta functions of eq. (5.1), the sigma model has off-shell divergences (proportional to the equation of motion $\nabla^2 X^\mu$) which are usually ignored because they can be changed by field redefinitions. If they are kept, a new beta function, amounting to a coordinate condition, is introduced and complete parallelism between the beta function and BRST equations is established. Because it can be changed by field redefinitions, this gauge-fixing beta function is essentially arbitrary and can be chosen for convenience.

If we use this freedom to make the linearized gauge choice

$$\partial^\nu h_{\nu\mu} - \tfrac{1}{2}\partial_\mu h^\nu_\nu = \partial_\mu \phi\,, \tag{5.3}$$

and expand eq. (5.1) to lowest order in weak fields, we obtain the system

$$\nabla^2 h_{\mu\nu} = -\tfrac{1}{2}\kappa\, e^{-\bar{\phi}/2}\eta_{\mu\nu},$$

$$\nabla^2 \phi = \tfrac{1}{4}\kappa\, e^{-\bar{\phi}/2}(D+2), \tag{5.4}$$

where D is the spacetime dimension. For comparison, the equations of motion and gauge condition extracted from BRST considerations in the previous section can be written in the similar, but not identical, form

$$\nabla^2 h_{\mu\nu} = -A\eta_{\mu\nu},$$

$$\nabla^2 \Phi = A,$$

$$\partial^\nu h_{\nu\mu} = \partial_\mu \Phi. \tag{5.5}$$

The loop normalization constant A takes on the value $2\lambda^2(N-32)$ for the superstring, but for the moment we want to emphasize the structure of the equations, rather than the particular theory-dependent values of constants. We have also used a different symbol for the dilaton fields appearing in the two equations in order to emphasize the final point, namely, that there is no guarantee that the curvature dilaton (ϕ) of the sigma model is the same as the ghost dilaton (Φ) of the string field theory approach. Thus, to show physical equivalence of the two sets of equations, it is only necessary to show that they are equivalent up to a field redefinition of one dilaton into the other.

In fact, it is easy to see that the identifications

$$\phi - \bar{\phi} = \Phi - \tfrac{1}{2}h^\mu{}_\mu,$$

$$A = \tfrac{1}{2}\kappa\, e^{-\bar{\phi}/2} \tag{5.6}$$

bring the two equations into coincidence. This guarantees that, at least to the order we are working, the loop-corrected BRST equations are internally consistent and derivable from a sensible covariant spacetime action. This is a very non-trivial check in the sense that if the relative coefficients of the graviton and dilaton in the loop anomaly (eq. (4.13)) had been any different, it would have failed. To demonstrate that the BRST method is totally equivalent to the divergence cancellation method, it is only necessary to verify that the loop normalization constants appearing in the two approaches satisfy the right relation. We have checked the bosonic case and find agreement.

The peculiar relation between the two versions of the dilaton could have been anticipated by an argument of Siegel's. If we start with a general bosonic string wave function (4.15), any change of that wave function of the form

$$|\delta_\varepsilon \Psi\rangle = (Q + \tilde{Q})|\varepsilon\rangle \tag{5.7}$$

for arbitrary $|\varepsilon\rangle$ is a gauge invariance of the free string action

$$\mathscr{L}_0 = \langle \Psi|(Q + \tilde{Q})|\Psi\rangle . \tag{5.8}$$

By choosing

$$|\varepsilon\rangle = \varepsilon_\mu(q)\left[\tilde{b}_{-1}\alpha^{\mu}_{-1} + b_{-1}\tilde{\alpha}^{\mu}_{-1}\right]|\downarrow\downarrow\rangle , \tag{5.9}$$

we generate the transformation

$$\delta_\varepsilon h_{\mu\nu} = \partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu ,$$

$$\delta_\varepsilon \Phi = \partial^\mu \varepsilon_\mu \tag{5.10}$$

on the string field in (4.15). As far as the metric goes, this is just a general coordinate transformation. That Φ is not invariant under this transformation indicates that it is not a general coordinate scalar and cannot be identified with the standard sigma model dilaton, which very definitely is a scalar. On the other hand, the combination

$$\phi - \bar{\phi} = \Phi - \tfrac{1}{2} h^{\mu}{}_{\mu} \tag{5.11}$$

is invariant and can be identified with the varying part of the curvature dilaton. But this is precisely the relation we found by comparing the two types of equation of motion. The same relations hold for the superstring, if we replace the gauge (5.9) by

$$|\varepsilon\rangle = \varepsilon_\mu(q)\left[\tilde{\beta}_{-1/2} d^{\mu}_{-1/2} - \beta_{-1/2}\tilde{d}^{\mu}_{-1/2}\right]|\downarrow\downarrow\rangle \tag{5.12}$$

This gauge transformation again leads to eqs. (5.10), and (5.11) is again a scalar.

The importance of the relation (5.11) can also be seen by directly comparing actions, rather than equations of motion. If we expand the spacetime action derived from the sigma model, eq. (5.2), in powers of Φ and $h_{\mu\nu}$, (assuming the gauge (5.3) and using (5.11) to eliminate the standard dilaton, ϕ) we get

$$S = \int \mathrm{d}^D X \mathrm{e}^{\bar{\phi}} \left\{ -\tfrac{1}{4}\left(h_{\mu\nu,\alpha}\right)^2 + \tfrac{1}{2}\left(\Phi_{,\alpha}\right)^2 + \cdots \right\} . \tag{5.13}$$

The wrong relative sign between the two kinetic energy terms is an indication that Φ is not a conventional scalar field. An alternate form of the action, appropriate to the BRST approach, is given by the free string action of eq. (5.8). It can be expanded as a functional of Φ and $h_{\mu\nu}$ using eq. (4.15) and, with due care in evaluating matrix elements of ghost creation and annihilation operators, shown to give the same result as eq. (5.13).

The bottom line of this discussion is that the loop-corrected equations of motion for background fields obtained by our new method of cancelling BRST anomalies between trees and loops is, to the order that we have calculated things, equivalent to loop-corrected beta functions obtained by cancelling divergences between trees and loops. It is perhaps worth pointing out that the consistency checks carried out here are more stringent than those previously done because our new technique allows us to independently evaluate the dilaton and graviton equations of motion. Any change of the relative coefficient between eqs. (4.25) would render a consistent solution impossible. We have every reason to believe, but as yet no evidence, that the two methods are equivalent to higher orders in the various expansion parameters as well.

6. Sigma model renormalization by a dilute gas of holes

In the previous sections, we showed that anomalies associated with the insertion of a *single* hole (open string loop) on the tree world sheet could be cancelled by the insertion of a single local operator. Interpreting the operator as a shift of the underlying sigma model gives a set of first string loop corrections to the stability equations for the spacetime background fields. The insertion of multiple independent holes, in a sort of dilute gas approximation, turns out not to modify these equations since the higher string loop corrections are encoded in the correlations induced by the overlap of holes with each other. Nevertheless, a treatment of the "dilute gas of holes" approximation brings out some interesting facts about the rôle of ghost vacua and the ghost dilaton vertex operator.

Consider then the usual tree amplitude for M closed strings in arbitrary excited states $|N_j\rangle$, $j=1,\dots,m$ emitted from a plane world sheet. The jth external particle is emitted at z_j, where it couples by a local current $V(z_j,\bar z_j;N_j)$, made of X^μ and its derivatives. The precise form of V depends on the excitation state N_j. The integrand of the tree amplitude is

$$\langle 0|\prod_{j=1}^{M} V(z_j,\bar z_j;N_j)|0\rangle\,. \tag{6.1}$$

Three of the z_j are left unintegrated by virtue of the usual Koba-Nielsen or SL(2, C)-invariance arguments. When ghosts are included in the proper way [32], the three unintegrated points have extra factors $c(z_j)c(\bar z_j)$, which are responsible for part of the measure (see appendix C for details).

Now convert the tree amplitude to a higher-loop amplitude by sewing cylinders ending in holes or crosscaps to some of the external lines of the sphere. The attachments have the form

$$\langle N_j|\left[L_0+\tilde L_0-2\right]^{-1}|B\rangle\,. \tag{6.2}$$

The state at the left must match the external line of the tree graph so that it can be attached by unitarity

$$\sum_{N_j} |N_j\rangle\langle N_j| = 1. \tag{6.3}$$

The cylinder has its usual closed string propagator with necessary ghost insertions (4.8) and ends in a boundary $|B\rangle$ or crosscap $|C\rangle$.

However, as discussed in appendix C, the plane and cylinder are built on different ghost vacua and it is important to match them correctly. If z_j is a moving (integrated) Koba-Nielsen point, the tree states will be built on the SL(2, C) invariant vacuum $\langle 0|$. As in eq. (C.6), the projection operator (6.3) must contain the bra with nonvanishing matrix element,

$$\langle N_j'| \sim \langle 0| c_{-1}\tilde{c}_{-1}c_0\tilde{c}_0\tilde{c}_1 c_1 . \tag{6.4}$$

To get a nonzero result, the attachment must thus contain the ghost factors

$$|N_j\rangle\langle N_j'| \dots |0\rangle = |N_j\rangle\langle N_j'| \dots \left(b_0 + \tilde{b}_0\right) b_{-1}\tilde{b}_{-1}|Z\rangle, \tag{6.5}$$

by eq. (C.18), where $|Z\rangle$ is the ghost vacuum appropriate for the cylinder. After sewing, we therefore get three antighost insertions. These correspond to the three Teichmüller parameters for a moving hole: its radius and the coordinates of the center [33]. Note that $\{Q, b_{-1}\} = L_{-1}$ generates translations.

On the other hand, if $z_j = 0$ is fixed, then by eq. (C.14), the tree states $|N_j\rangle$ are built on $c_1\tilde{c}_1|0\rangle$. We then get, by eq. (C.19),

$$|N_j\rangle\langle N_j'| \dots c_1\tilde{c}_1|0\rangle = |N_j\rangle\langle N_j'| \dots \left(b_0 + \tilde{b}_0\right)|Z\rangle. \tag{6.6}$$

In this case, the center of the attached hole is fixed at $z = \bar{z} = 0$, and only its radius varies. The single antighost insertion corresponds to the single Teichmüller parameter. Matching ghost vacua when sewing thus automatically generates the correct antighost insertions in all cases [34].

By eq. (4.2), the divergences and BRST anomalies of eq. (6.2) can be cancelled by counterterms

$$\langle N_j|\Psi\rangle = \langle N_j|\Psi(0)|0\rangle, \tag{6.7}$$

where $|\Psi\rangle$ is eq. (4.15) or something similar. Attaching these to the sphere by eq. (6.3) has the effect of making the replacement

$$V(z_j, \bar{z}_j; N_j) \to \Psi(z_j, \bar{z}_j). \tag{6.8}$$

in eq. (6.1). But the state $|\Psi\rangle$ can be created by the action of a local current on the sigma model vacuum: if it corresponds to a pure graviton, the appropriate current is

$$\Psi(z,\bar{z}) = :h_{\mu\nu}(X(z))\,\partial_{\bar{z}}X^{\mu}(z)\,\partial_z X^{\nu}(z):, \tag{6.9}$$

which is recognizable as a sigma model interaction term. If one shifts the sigma model and expands in powers of the interaction term, using the same structure of Koba-Nielsen fixed points and ghost insertions as used in the dilute gas of holes, then all BRST anomalies and divergences can be cancelled by the one-loop sigma model shifts derived in the previous chapter. The only subtlety is the question of Koba-Nielsen fixed points and ghost insertions to be used in expanding the sigma model with shifted interaction, but the prescription described above seems to be the unique consistent one.

There are some confusing issues which we have not completely sorted out. For the superstring, there are two forms of the graviton vertex operator, one fermionic (as in eq. (4.21)) and one bosonic (as in eq. (4.15)). In fact, both have to be used in calculating a given tree amplitude and crossing symmetry [30] is recovered by applying a picture-changing operation (4.24) which converts one vertex into the other. In the "F_2 picture," the rule is to use the fermionic vertex for the lines fixed at $z = 0, \infty$, and the bosonic form for all others [30]. The form of the vertex which corresponds directly to a sigma model interaction term is apparently that associated with moving Koba-Nielsen points. This is analogous to our discussion of the bosonic theory, where the difference between fixed and moving vertices lay in the presence or absence of extra ghost insertions.

The dilaton presents a similar problem. The dilaton piece of eq. (4.17) can be converted to a local operator insertion using eq. (C.4) with the result

$$\begin{aligned}\left[\tilde{c}_{-1}b_{-1} + c_{-1}\tilde{b}_{-1}\right]c_1\tilde{c}_1|0\rangle &= [\tilde{c}_{-1}\tilde{c}_1 - c_{-1}c_1]|0\rangle \\ &= \lim_{z\to 0}\left[c(z)\,\partial_z^2 c(z) - \tilde{c}(\bar{z})\,\partial_{\bar{z}}^2\tilde{c}(\bar{z})\right]|0\rangle .\end{aligned} \tag{6.10}$$

This identifies the operator insertion at a fixed point. To get the insertion at a moving (integrated) Koba-Nielsen point, one should strip off the $c_1\tilde{c}_1$ ghost insertions and reexpress the remainder as an equivalent local operator. The best we can do is

$$\tilde{c}_{-1}b_{-1} + c_{-1}\tilde{b}_{-1} \approx \lim_{z\to 0}\left\{\partial_{\bar{z}}^2\tilde{c}(\bar{z})\oint \mathrm{d}z'\,b(z') + \partial_z^2 c(z)\oint \mathrm{d}\bar{z}'\,\tilde{b}(\bar{z}')\right\}, \tag{6.11}$$

since b_{-1} is the coefficient of z^{-1} in eq. (C.4) and therefore cannot be represented by a local operator. In neither case do we get the ghost number current bc, $\tilde{b}\tilde{c}$, whose divergence is the world sheet curvature [35], and the connection to the curvature dilaton term of the sigma model action is not evident. We expect that a careful study of the behavior of these operators on a curved world sheet will resolve this issue. Possibly the picture-changing operators have additional factors, which contribute only to the pure ghost wave functions.

7. Discussion

It is becoming clear that string theories can be consistent without being finite graph by graph and that the class of consistent theories is much larger than the class of strictly finite theories. It is obviously important to develop our understanding of this new class of theories, since, for all we know, they are needed to describe the vacuum and other important states. The present paper is but a preliminary step in that direction, probably revealing more gaps than it fills. In our concluding paragraphs, we will list what seem to us the most important topics to pursue in further work.

In the covariant formalism, the closed superstring has four sectors: NS-NS and R-R contain bosons, while NS-R and R-NS contain fermions. We have only considered vacuum transitions in the NS-NS sector. The R-R sector has notorious technical problems from the superghost zero modes, but of course should not be ignored. We hope that the methods of appendix B will be helpful in this connection.

We need a systematic procedure for extending our renormalization scheme to higher string loop order. As discussed in sect. 5, higher-loop physics comes not from the multiple-hole diagrams directly, but rather from the new anomalies and divergences which arise when two or more holes collide. Giving a serious treatment of these effects and generating the associated higher-order loop corrections to the background field equations of motion is bound to be quite difficult.

We have not treated closed string loops in any detail, even though we expect the really interesting theories to contain *only* closed strings. This is simply because open strings are easier to deal with at the lowest order in string perturbation theory, and we regard the open string theory as a convenient testing ground for our picture of string renormalization theory. Adding a handle, or "lifebelt", to a tree world sheet is more complicated than adding a boundary, because the vacuum transition amplitude can have complicated internal structure in the presence of background fields. We have hopes that our new BRST anomaly cancellation method will make this very interesting case tractable also.

We only considered the first-order beta functions of the sigma model. It should be possible to reconstruct the beta function to any order of perturbation theory by requiring that the perturbed BRST operator annihilate the perturbed wave function. Carrying this out explicitly should throw considerable light on the field theory of closed strings.

Finally, although we used BRST arguments to derive our loop-corrected equations for vacuum stability, we did not have anything to say about the most important role of BRST invariance in usual string applications, namely, the decoupling of negative metric states. In fact, since the loop corrections take the sigma model away from its standard BRST-invariant form, one is guaranteed that the vertex operators corresponding to negative metric particles will no longer decouple. Presumably, the allowed vertex operators suffer loop corrections along with the background fields, and it remains true that the loop-level negative metric particles still decouple. We have only very vague ideas about how to explore this important matter, but are convinced that keeping BRST ideas in the forefront will be helpful here too.

In a related vein, we have to note that, once the sigma model loses conformal invariance, amplitudes will depend on the conformal scale factor of the two-dimensional world sheet metric, not just on the usual moduli. Technically speaking, the Polyakov path integral becomes slice-dependent and the standard picture of string perturbation theory as an integral over densities on moduli space breaks down. Although this might not be a disaster, it would certainly be awkward. We believe that the loop-corrected beta functions derived in this paper and its predecessor are precisely the condition that the slice dependence of the tree amplitude (induced by the conformally non-invariant sigma model) cancel against the slice dependence of the loop amplitude (induced by the conformal anomaly at the boundary of moduli space). If this is so, then the loop-corrected beta functions do serve to impose a generalized version of conformal invariance, realized not on individual two-dimensional world sheets, but on all world sheets at once. The new string theories we are constructing should be associated with a new class of conformal sigma models. We hope to pursue this, and the other above-mentioned topics, in future publications.

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Appendix A

THE MÖBIUS STRIP

Here we construct the Möbius strip amplitude of the bosonic string as the scalar product of the eigenvectors for a boundary of unit radius and a crosscap of radius r (see fig. 2).

If M tachyons of momenta k_j enter at $(\sigma, \tau) = (\sigma_j, 0)$, then the boundary condition is

$$\frac{1}{4\pi i}\frac{\partial X^\mu}{\partial \tau}\bigg|_{\tau=0} = \sum_{j=1}^{M} k_j^\mu \delta(\sigma - \sigma_j)\,. \tag{A.1}$$

Equating coefficients of $e^{im\sigma}$ as in (2.4) now gives

$$p=\sum_{j=1}^{M} k_j=0,$$

$$a_m^\dagger-\tilde{a}_m=-\frac{i}{\sqrt{m}}\sum_{j=1}^{M} k_j e^{-im\sigma_j}\equiv f_m,$$

$$\tilde{a}_m^\dagger-a_m=-\frac{i}{\sqrt{m}}\sum_{j=1}^{M} k_j e^{im\sigma_j}\equiv \tilde{f}_m. \tag{A.2}$$

The left eigenvector is therefore

$$\langle 0|\exp\left\{\sum_{m=1}^{\infty}\left[a_m\cdot\tilde{a}_m+f_m\cdot a_m+\tilde{f}_m\cdot\tilde{a}_m\right]\right\}. \tag{A.3}$$

We multiply this by a crosscap operator (2.16) at $r=e^{\tau}$:

$$\exp\left\{\sum_{m=1}^{\infty}(-r^2)^m a_m^\dagger\cdot\tilde{a}_m^\dagger\right\}|0\rangle. \tag{A.4}$$

For a single pair of oscillators,

$$\langle 0|e^{u\tilde{a}a+fa+\tilde{f}\tilde{a}}e^{va^\dagger\tilde{a}^\dagger}|0\rangle=[1-uv]^{-1}\exp\{f\tilde{f}v[1-uv]^{-1}\}. \tag{A.5}$$

The scalar product (A.3) × (A.4) is then

$$\prod_{n=1}^{\infty}[1-(-r^2)^n]^{-D}\exp\left\{\sum_{m=1}^{\infty}f_m\tilde{f}_m(-r^2)^m[1-(-r^2)^m]^{-1}\right\}. \tag{A.6}$$

We rearrange the exponent by expanding in r, inserting (A.2), and then summing over m to get

$$\prod_{n=1}^{\infty}[1-(-r)^n]^{-D}\times\prod_{i,j=1}^{M}\left\{\prod_{n=1}^{\infty}\left[1-e^{i(\sigma_i-\sigma_j)}(-r^2)^n\right]\right\}^{k_i\cdot k_j}. \tag{A.7}$$

Now we determine the integration measure for r and σ_i. The closed string propagator is

$$[L_0+\tilde{L}_0-2]^{-1}=\int_0^1\frac{dr}{r^3}r^{L_0+\tilde{L}_0}. \tag{A.8}$$

We can scale the last factor of (A.8) out of (A.4), so the r measure must be $\int_0^1 dr/r^3$. Multiplying (A.3) by $|0\rangle$ should give the Koba-Nielsen amplitude for a disc [20], which requires the σ_i measure to be a constant times

$$\int_0^{2\pi} \prod_{i=1}^{M-1} d\sigma_i\, \theta(\sigma_{i+1} - \sigma_i) \prod_{i<j} \left[\sin\left(\tfrac{1}{2}\sigma_j - \tfrac{1}{2}\sigma_i\right)\right]^{k_i \cdot k_j}. \tag{A.9}$$

Combining (A.7) and (A.9) leads easily to the usual amplitude for the Möbius strip [36]. Note that the latter was obtained by sewing together the ends of a twisted *open* string. The agreement with a *closed* string transition amplitude is nontrivial.

Appendix B

GHOST FACTORIZATION

Here we factorize the partition function of the open string ghosts in the closed string channel. It is instructive to couple a chemical potential μ to the ghost number U. We will show that

$$\mathrm{tr}\left(w^{H_O} e^{i\pi\mu U_O}\right) = r^{(\mu-1)^2/4} \langle B' | r^{(\mu-1)(U_L - U_R)/2} r^{H_L + H_R} | B \rangle. \tag{B.1}$$

Here H_O is the hamiltonian and U_O is the ghost number of the *open* string, while H_L, U_L and H_R, U_R are the corresponding operators for the left- and right-moving modes of the *closed* string. Each of these can be written in terms of fermion oscillators with $\{b_m, c_{-n}\} = \delta_{mn}$. We have [37]

$$H = \sum_{n=1}^{\infty} n(b_{-n}c_n + c_{-n}b_n) + \tfrac{1}{12}, \tag{B.2}$$

$$U = \tfrac{1}{2}[c_0, b_0] + \sum_{n=1}^{\infty} (c_{-n}b_n - b_{-n}c_n). \tag{B.3}$$

Each can also be written in a bosonized form [37]

$$H = \tfrac{1}{2}p^2 + \sum_{n=1}^{\infty} \phi_{-n}\phi_n - \tfrac{1}{24}, \tag{B.4}$$

$$U = p, \tag{B.5}$$

where $[\phi_m, \phi_{-n}] = m\delta_{mn}$ and p is a "momentum" with half-integer eigenvalues. (So we have three distinct sets of oscillators O, L, R, each of which can be either fermions or bosons.) In eq. (B.1), $\langle B'|$ and $|B\rangle$ are the closed string boundary operators, and

$$r = e^{2\pi^2/\log w}. \tag{B.6}$$

The left-hand side of (B.1) can now be evaluated:

$$\mathrm{tr}\left(w^{H_0} e^{i\pi\mu U_0}\right) = w^{-1/24}\vartheta_2\left(\tfrac{1}{2}\mu, w^{1/2}\right)\prod_{m=1}^{\infty}\left(1-w^m\right)^{-1}, \tag{B.7}$$

where ϑ_2 is an elliptic function. The product and sum formulae for ϑ_2 generate the partition functions of (B.2)–(B.3) and of (B.4)–(B.5) respectively [37].

Applying the Jacobi transformation (B.6) to ϑ_2, we get two more formulae:

$$(\mathrm{B}.7) = r^{-1/12+\mu^2/4}\left(1-r^{1-\mu}\right)\prod_{m=1}^{\infty}\left(1-r^{2m+1-\mu}\right)\left(1-r^{2m+\mu-1}\right), \tag{B.8}$$

$$(\mathrm{B}.7) = r^{-1/3+\mu^2/4}\sum_{n=-\infty}^{\infty}(-1)^n r^{(n-1/2)^2+n(1-\mu)}\prod_{m=1}^{\infty}\left(1-r^{2m}\right)^{-1}. \tag{B.9}$$

These must give the fermionic and bosonic forms of the right-hand side of (B.1), respectively. By eqs. (3.20) and (3.22), the fermionic boundary operators of the closed string ghosts are (recall that ~ means right-moving)

$$|B\rangle = \exp\left\{\sum_{m=1}^{\infty}\left[c_{-m}\tilde{b}_{-m} + \tilde{c}_{-m}b_{-m}\right]\right\}(c_0+\tilde{c}_0)|\downarrow\downarrow\rangle, \tag{B.10}$$

$$\langle B'| = \langle\uparrow\uparrow|\left(b_0-\tilde{b}_0\right)\exp\left\{\sum_{m=1}^{\infty}\left[\tilde{b}_m c_m + b_m\tilde{c}_m\right]\right\}. \tag{B.11}$$

Inserting (B.2), (B.3), (B.10), and (B.11) into the right-hand side of (B.1) indeed gives (B.8).

The bosonic form of the ghost boundary operators is

$$|B\rangle = \sum_{n=-\infty}^{\infty} e^{i(n-1/2)(q_L-q_R)}\exp\left\{\sum_{m=1}^{\infty}\frac{1}{m}\phi_{-m}\tilde{\phi}_{-m}\right\}|0\rangle, \tag{B.12}$$

$$\langle B'| = \langle 0|\exp\left\{\sum_{m=1}^{\infty}\frac{1}{m}\phi_m\tilde{\phi}_m\right\}\sum_{n=-\infty}^{\infty}(-1)^{n+1}e^{-i(n-1/2)(q_L-q_R)}, \tag{B.13}$$

where q_L and q_R are the "positions" conjugate to the left and right ghost numbers p_L and p_R. Inserting (B.4), (B.5), (B.12), and (B.13) into the right-hand side of (B.1) indeed gives (B.9). Thus (B.1) is an operator representation of the Jacobi transform (B.7) = (B.8) = (B.9).

The actual ghost partition function has $\mu = 1$. The factor $(-1)^{U_O}$ in (B.1) imposes periodic boundary conditions on the Faddeev-Popov determinant [37, 38, 11]. However, (B.8) vanishes at $\mu = 1$ unless we take a derivative $\partial/\partial\mu$ first. This brings down a factor $U_L - U_R$ containing the term $(c_0 - \tilde{c}_0)(b_0 + \tilde{b}_0)$ needed (by eqs. (4.7)–(4.8)) to mop up the ghost zero modes on the cylinder [24].

These formulae give considerable information on the coupling of open and closed strings [39]. Open string states with all ghost numbers U_O can flow around the annulus. This becomes essentially the difference $U_L - U_R$, which is not conserved at the boundary. If the closed string is to vanish into the vacuum, $U_L + U_R$ must be 0 (not -1).

Appendix C

GHOST VACUA

This appendix elucidates some confusing technical details, understood by experts but not all readily available.

The plane is SL(2, C) invariant. Therefore closed string tree graphs must be built on a ghost vacuum $|0\rangle$ which is annihilated by $L_{\pm 1}$, L_0, $\tilde{L}_{\pm 1}$, $\tilde{L}_0$. This requires [24]

$$\begin{aligned} c_n|0\rangle &= 0, \qquad n \geqslant 2, \\ b_n|0\rangle &= 0, \qquad n \geqslant -1. \end{aligned} \tag{C.1}$$

Since $c_n^\dagger = c_{-n}$, the adjoint satisfies

$$\begin{aligned} \langle 0|c_n &= 0, \qquad n \leqslant -2, \\ \langle 0|b_n &= 0, \qquad n \leqslant 1. \end{aligned} \tag{C.2}$$

The right modes $\tilde{c}_n$, $\tilde{b}_n$ obey similar equations. Since

$$\{b_m, c_{-n}\} = \delta_{mn} \tag{C.3}$$

the Fermi levels of $|0\rangle$ and $\langle 0|$ do not match, and they are orthogonal. To get a nonzero matrix element, three c's and three $\tilde{c}$'s must be inserted [24]. It is very convenient to associate these with the three fixed Koba-Nielsen points z_R, z_S, z_T. Because c, b have conformal weight -1, $+2$, the mode expansions on the plane are

$$\begin{aligned} c(z) &= \sum_n c_n z^{-n+1}, \\ b(z) &= \sum_n b_n z^{-n-2}, \end{aligned} \tag{C.4}$$

and therefore

$$\langle 0|c(z_R)c(z_S)c(z_T)|0\rangle = (z_R - z_S)(z_S - z_T)(z_T - z_R)\langle 0|c_{-1}c_0c_1|0\rangle \quad (C.5)$$

and similarly for $\tilde{c}(\bar{z})$. The nonvanishing terms thus generate the fixed point measure [1]. So the rule is to associate $c(z)\tilde{c}(\bar{z})$ with each fixed Koba-Nielsen point z, but not with the integrated moving ones.

It also follows that the projection operator onto $|0\rangle$ will be

$$\mathscr{P} = |0\rangle\langle 0|c_{-1}\tilde{c}_{-1}c_0\tilde{c}_0\tilde{c}_1c_1\,, \quad (C.6)$$

since otherwise we could not have $\mathscr{P}^2 = \mathscr{P}$.

Now we consider the cylinder. We can map it onto a plane with two punctures by

$$\xi = \log z\,. \quad (C.7)$$

A quantity Y with conformal weight h transforms by

$$Y(\xi) = \left(\frac{dz}{d\xi}\right)^h Y(z)\,. \quad (C.8)$$

Since c, b have $h = -1, +2$, eqs. (C.4) thus become

$$c(\xi) = \sum_n c_n e^{-n\xi}\,,$$

$$b(\xi) = \sum_n b_n e^{-n\xi}\,. \quad (C.9)$$

The cylinder is only invariant under rotations, so there is no reason to use the SL(2, C) vacuum (C.1). The boundary operator (3.20) shows that the correct vacuum is

$$|Z\rangle = (c_0 + \tilde{c}_0)|\downarrow\downarrow\rangle\,. \quad (C.10)$$

Here $|\downarrow\downarrow\rangle$ is the Siegel vacuum, commonly used in string field theory, and satisfying

$$b_n|\downarrow\downarrow\rangle = 0\,, \qquad n \geqslant 0\,,$$

$$c_n|\downarrow\downarrow\rangle = 0\,, \qquad n \geqslant 1\,, \quad (C.11)$$

and similarly for the right modes. Since we are dealing with fermions, (C.11) and (C.1) must be related by

$$|\downarrow\downarrow\rangle = c_1\tilde{c}_1|0\rangle \quad (C.12)$$

(up to a sign convention), and thus

$$|Z\rangle = (c_0 + \tilde{c}_0) c_1 \tilde{c}_1 |0\rangle . \tag{C.13}$$

Notice that on the plane by (C.4)

$$|\downarrow\downarrow\rangle = c(0)\tilde{c}(0)|0\rangle . \tag{C.14}$$

So the fixed Koba-Nielsen insertions at $z = 0$ almost convert the SL(2, C) vacuum to a cylinder vacuum.

By (3.22) the left cylinder vacuum is

$$\langle Z| = \langle\uparrow\uparrow|(b_0 - \tilde{b}_0) , \tag{C.15}$$

where

$$\langle\uparrow\uparrow|b_n = 0 , \qquad n \leqslant -1 ,$$

$$\langle\uparrow\uparrow|c_n = 0 , \qquad n \leqslant 0 , \tag{C.16}$$

and therefore by (C.11)

$$\langle\uparrow\uparrow|\downarrow\downarrow\rangle = 1 . \tag{C.17}$$

Eqs. (C.13) and (C.3) lead to the important equations for transforming vacua when sewing a cylinder onto a sphere:

$$|0\rangle = (b_0 + \tilde{b}_0) b_{-1} \tilde{b}_{-1} |Z\rangle , \tag{C.18}$$

$$\tilde{c}_1 c_1 |0\rangle = (b_0 + \tilde{b}_0)|0\rangle . \tag{C.19}$$

The antighosts here become the Teichmüller zero modes of the functional integral formalism [33].

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