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OPEN STRINGS IN BACKGROUND GAUGE FIELDS

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Some properties of open bosonic strings in a background abelian gauge field are investigated. We derive the equations of motion and effective action for the electromagnetic field, and discuss the effect of the field on the string spectrum and partition function.

1. Introduction

The study of strings propagating in background fields has yielded interesting results in the case of closed strings, where the conformal invariance has been used to derive the variational equations of a spacetime effective action for the background fields [1–6]. One purpose of this paper is to find the equations of motion of a background gauge field in a theory of open bosonic strings. We will deal only with abelian gauge fields, since the nonabelian case is somewhat more complicated. Moreover, we will work in the approximation of slowly varying fields. In this approximation, our result is exact to all orders in the inverse string tension α'. Our method involves computing the beta function for the electromagnetic field at the open-string tree level. Conformal invariance implies the vanishing of the beta function, which gives the equations of motion for the field strength $F_{\mu\nu}$. From these equations, we can derive an effective action. Fradkin and Tseytlin [7] have obtained an effective action for an abelian gauge field coupled to open bosonic strings by a totally different procedure, using the Polyakov path integral. In the limit of a constant field, the path integral reduces to a gaussian integral over fields on the boundary, and can be exactly computed. Interestingly enough, the two procedures yield the same action (just as they did in the closed string case) even though it is not a priori obvious that they should*.

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* The basic issue is that the beta function is associated with the one-particle-irreducible generating functional while the Polyakov path integral naturally produces the connected generating functional. The single-massless-particle poles in the connected functional lead to divergences in the path integral which must be regulated. The most satisfying situation would be to find a unique regulator which, in effect, truncated the connected functional to the one-particle-irreducible functional. Steps in this direction have recently been taken by several authors [8–10].

An effective action for $F_{\mu\nu}$ has also been reconstructed up to terms quartic in F by calculating four-point scattering amplitudes, both in the case of the open bosonic string [10] and in the case of the open superstring [11]. The result is the same in both cases, and is compatible with ours. However, since our result is valid to all orders in α', one would have to sum the contributions of infinitely many string tree graphs to obtain it using the S-matrix approach.

The method of extracting the effective action from the beta function cannot easily be generalized to include the effect of string loops, since the beta function is independent of the world-sheet topology. In an attempt to get some understanding of the string loop corrections in the presence of a background field, we investigate the vacuum-to-vacuum amplitude on the annulus, using two different methods. The first, based on the stress-energy tensor, requires computing the exact propagator on the annulus in the presence of a gauge field. The other, based on the operator formalism, seems to yield more information about the dependence of the partition function on the background field. This method requires computing the spectrum of the open string in a background gauge field, an analysis which is interesting on its own merits. The partition function on the annulus in the presence of gauge fields has also been computed by Fradkin and Tseytlin [7], using path integral methods. Our result agrees with the divergent part of their result.

This paper is organized as follows. In sect. 2, we derive the equations of motion for the electromagnetic field, i.e. the string modification to the Maxwell equations. This calculation is performed at the string tree level, using background field methods. We show that this equation can be obtained by varying the Born-Infeld lagrangian [7,12,13], which is therefore an effective lagrangian for the electromagnetic field, at the string tree level. In sect. 3, we move to the one-loop level and compute the Neumann propagator on the annulus when the boundary condition is modified by the presence of gauge fields. This result is then used in sect. 4 to compute the partition function on the annulus. In sect. 5, we find the spectrum of the bosonic open string a constant background electromagnetic field. We study both the case of strings with zero net charge and with nonzero net charge. The results are used in sect. 6 to compute again the partition function on the annulus using the operator formalism. Finally, in appendix A, we give an alternate derivation of the beta function for $F_{\mu\nu}$, and in appendix B, we discuss the effect of a toroidal compactification on open strings in a magnetic field.

2. Modified Maxwell equations and effective action

In this section we will compute the equation of motion for an abelian gauge field coupled to the open Bose string. We will work at the tree level in string theory and represent the string world-sheet with the upper half-plane as in fig. 1. Throughout this section we will adopt euclidean metrics both on the world sheet and in spacetime.

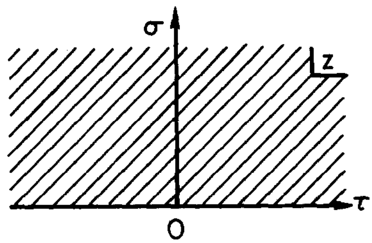

Fig. 1. Upper half-plane representation of the string tree-level world sheet.

The action for the string coupled on the boundary to the electromagnetic field is

$$S=\frac{1}{2\pi\alpha'}\left[\tfrac{1}{2}\int_{\mathrm{M}^2}\mathrm{d}^2z\,\partial^a X_\mu\,\partial_a X^\mu+i\int_{\partial\mathrm{M}}\mathrm{d}\tau A_\mu\,\partial_\tau X^\mu\right], \tag{2.1}$$

where A^μ has been rescaled to contain a factor $2\pi\alpha'$. In this calculation we will use the background field approach [4,14]. If we expand the action (2.1) around an arbitrary background $\bar{X}$, i.e.

$$X^\mu(\tau,\sigma)=\bar{X}^\mu(\tau,\sigma)+\xi^\mu(\tau,\sigma), \tag{2.2}$$

we get

$$\begin{aligned}S[\bar{X}+\xi]=S[\bar{X}]+\frac{1}{2\pi\alpha'}\Bigg[&\int_{\mathrm{M}^2}\mathrm{d}^2z\left(\partial^a\bar{X}_\mu\,\partial_a\xi^\mu+\tfrac{1}{2}\partial^a\xi_\mu\,\partial_a\xi^\mu\right)\\&+i\int_{\partial\mathrm{M}}\mathrm{d}\tau\Big(F_{\mu\nu}\xi^\mu\,\partial_\tau\bar{X}^\nu+\tfrac{1}{2}\nabla_\nu F_{\mu\lambda}\xi^\nu\xi^\lambda\,\partial_\tau\bar{X}^\mu\\&+\tfrac{1}{2}F_{\mu\nu}\xi^\nu\,\partial_\tau\xi^\mu+\tfrac{1}{3}\nabla_\nu F_{\mu\lambda}\xi^\nu\xi^\lambda\,\partial_\tau\xi^\mu+\cdots\Big)\Bigg],\end{aligned} \tag{2.3}$$

where $\nabla_\nu=\partial/\partial X^\nu$ and $F_{\mu\nu}=\nabla_\mu A_\nu-\nabla_\nu A_\mu$. Here we will be working in the approximation of slowly varying fields. Therefore we will neglect terms with more than one derivative of F.

On-shell, the terms linear in ξ disappear since $\bar{X}$ satisfies the equations of motion

$$\begin{aligned}&\Box\bar{X}^\mu=0,\\&\partial_\sigma\bar{X}^\mu+iF^\mu_\nu\partial_\tau\bar{X}^\nu|_{\partial\mathrm{M}}=0,\end{aligned} \tag{2.4}$$

where $\Box = \partial_\tau^2 + \partial_\sigma^2$. Therefore the on-shell action is more simply

$$S[\bar{X}+\xi] = S[\bar{X}] + \frac{1}{2\pi\alpha'}\int_{\mathrm{M}^2} \mathrm{d}^2 z\, \tfrac{1}{2}\partial^a \xi_\mu\, \partial_a \xi^\mu$$
$$+ \frac{i}{2\pi\alpha'}\int_{\partial\mathrm{M}} \mathrm{d}\tau \left(\tfrac{1}{2}\nabla_\nu F_{\mu\lambda}\xi^\nu \xi^\lambda\, \partial_\tau \bar{X}^\mu + \tfrac{1}{2}F_{\mu\nu}\xi^\nu\, \partial_\tau \xi^\mu + \tfrac{1}{3}\nabla_\nu F_{\mu\lambda}\xi^\nu\xi^\lambda\, \partial_\tau \xi^\mu + \cdots \right). \tag{2.5}$$

We want to compute the one-loop (in the field theory sense) counterterm to the gauge coupling term in eq. (2.1), namely a counterterm of the form

$$\Delta S_I[\bar{X}] = \frac{i}{2\pi}\int_{\partial\mathrm{M}} \mathrm{d}\tau\, \Gamma_\mu \partial_\tau \bar{X}^\mu. \tag{2.6}$$

From Γ_μ we will extract the beta function β^A for the electromagnetic field A_μ. By imposing conformal invariance, i.e. $\beta^A = 0$, we will derive the equations of motion for $F_{\mu\nu}$.

The Neumann propagator in the upper half-plane satisfies the equations

$$\frac{1}{2\pi\alpha'}\Box G(z,z') = -\delta(z-z'),$$
$$\partial_\sigma G(z,z')\big|_{\sigma=0} = 0, \tag{2.7}$$

where $z = \tau + i\sigma$. The solution is

$$G(z,z') = -\alpha'(\ln|z-z'| + \ln|z-\bar{z}'|). \tag{2.8}$$

To compute the counterterm (2.6) we could work with this propagator and sum up all one-loop graphs with an external $\partial_\tau \bar{X}$ and all possible insertions of the vertex $F_{\mu\nu}\xi^\nu\partial_\tau\xi^\mu$. This calculation is carried out in appendix A. A more straightforward method is to compute the exact propagator in the presence of the gauge field F. In the presence of the gauge fields the propagator must satisfy the following boundary condition:

$$\partial_\sigma G(z,z')_{\mu\nu} + iF_\mu^{\ \lambda}\partial_\tau G_{\lambda\nu}(z,z')\big|_{\sigma=0} = 0. \tag{2.9}$$

We find that the solution to this equation is

$$G_{\mu\nu}(z,z') = -\alpha'\left[\delta_{\mu\nu}\ln|z-z'| + \frac{1}{2}\left(\frac{1-F}{1+F}\right)_{\mu\nu}\ln(z-\bar{z}') + \frac{1}{2}\left(\frac{1+F}{1-F}\right)_{\mu\nu}\ln(\bar{z}-z')\right]. \tag{2.10}$$

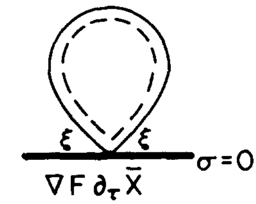

Fig. 2. The only one-loop graph contributing to β_A. The line ==== indicates the exact F-dependent propagator.

This is symmetric under the exchange $z \leftrightarrow z'$ and $\mu \leftrightarrow \nu$. For $F = 0$ this propagator reduces to the one of eq. (2.8).

The only counterterm to S_I is given by the one-loop graph in fig. 2. The evaluation of this graph gives

$$\Delta S_I = \frac{-i}{2\pi\alpha'} \int d\tau \, \tfrac{1}{2} \nabla_\nu F_{\mu\lambda} \, \partial_\tau \bar{X}^\mu G^{\nu\lambda}(\tau, \tau')|_{\tau \to \tau'}, \tag{2.11}$$

where $G^{\nu\lambda}(\tau, \tau')$ is the propagator on the boundary ($\sigma = \sigma' = 0$). In the limit $\tau \to \tau'$ we get

$$G_{\nu\lambda}(\tau \to \tau') = -\alpha' \left[1 + \frac{1}{2} \frac{1-F}{1+F} + \frac{1}{2} \frac{1+F}{1-F} \right]_{\nu\lambda} \ln \Lambda = -2\alpha' \ln \Lambda (1 - F^2)^{-1}_{\nu\lambda}, \tag{2.12}$$

where Λ is a short-distance cutoff. The beta function can now be obtained by differentiating with respect to the cutoff Λ,

$$\beta_\mu^A = \Lambda \frac{\partial}{\partial \Lambda} \Gamma_\mu = \nabla^\nu F_\mu^{\ \lambda} (1 - F^2)^{-1}_{\lambda\nu}. \tag{2.13}$$

The equations of motion are therefore

$$\nabla^\nu F_\mu^{\ \lambda} (1 - F^2)^{-1}_{\lambda\nu} = 0. \tag{2.14}$$

We should remember here that F is actually $2\pi\alpha' F$ and therefore eq. (2.14) contains all orders in α'. We have obtained the exact beta function to lowest order in derivatives of F. Graphs with two or more loops would introduce corrections of a higher order in derivatives. At the leading order in α', eq. (2.14) reduces to Maxwell's equations. Notice that, since F is a real antisymmetric tensor, F^2 has only negative eigenvalues, and hence eq. (2.14) makes sense.

A remark is in order. The propagator in eq. (2.10) can also be written as

$$G_{\mu\nu}(z, z') = -\alpha' \left[\ln|z - z'| + \frac{1 + F^2}{1 - F^2} \ln|z - \bar{z}'| - \frac{F}{1 - F^2} \ln \frac{z - \bar{z}'}{\bar{z} - z'} \right]_{\mu\nu}. \tag{2.15}$$

This rewriting shows that we get an F-dependent logarithmic divergence from the propagator (2.10) and hence a F-dependent contribution to the beta function only when $z \to z'$ on the boundary of the world sheet. Suppose in fact that $\sigma = \sigma' \neq 0$. Then the only logarithmic divergence comes from $\ln|z - z'|$ in eq. (2.15), while the F-dependent term $\ln|z - \bar{z}'|_{z \to z'} = \ln(2\sigma)$ is finite and gives no contribution to the beta function. This means that the beta function for the closed string fields (for instance for the graviton and dilaton fields) are not affected by the presence of gauge fields on the boundary.

We have computed the string correction to Maxwell equations and obtained eq. (2.14). The obvious question now is to find the action whose variation gives these equations. The answer is that although the beta function (2.13) is not the variation of any action, it is possible to exhibit an action whose variation is $\chi_{\mu\nu}\beta_A^\nu$, where $\chi_{\mu\nu}(F)$ is an invertible tensor. Therefore, the variational equation is equivalent to $\beta_A = 0$. A general relation of this form between the effective action and beta functions is suggested in [9, 15], and a special case of this relation has been found to occur for closed strings in [6]. Let us first notice that the following identity holds:

$$(1 - F^2)^{-1}_{\mu\nu}\beta_A^\nu = \nabla^\nu\left(\frac{F}{1 - F^2}\right)_{\mu\nu} - \left(\frac{F}{1 - F^2}\right)_{\mu\lambda}\nabla^\nu F^{\lambda\rho}\left(\frac{F}{1 - F^2}\right)_{\rho\nu}. \tag{2.16}$$

The second term in the r.h.s. of the above equation can be rewritten by making use of the antisymmetry of F and of the Bianchi identity. Then eq. (2.16) becomes

$$(1 - F^2)^{-1}_{\mu\nu}\beta_A^\nu = \nabla^\nu\left(\frac{1}{1 - F^2}\right)_{\mu\nu} + \frac{1}{4}\left(\frac{F}{1 - F^2}\right)_{\mu\nu}\nabla^\nu \operatorname{tr}\ln(1 - F^2). \tag{2.17}$$

This suggests that our equation of motion might be derived from an action that is a function of $\operatorname{tr}\ln(1 - F^2)$. Along these lines one can indeed derive the lagrangian

$$\mathscr{L}_{\text{eff}} = \exp\left(\tfrac{1}{4}\operatorname{tr}\ln(1 - F^2)\right) = \exp\left(\tfrac{1}{2}\operatorname{tr}\ln(1 + F)\right) = \sqrt{\det(1 + F)}\,, \tag{2.18}$$

whose Euler-Lagrange equations are

$$\sqrt{\det(1 + F)}\,(1 - F^2)^{-1}_{\mu\nu}\beta_A^\nu = 0. \tag{2.19}$$

Notice that this equation has the same solutions as $\beta_\nu^A = 0$. The Born-Infeld action $\sqrt{\det(1 + F)}$ is exactly the one obtained by Fradkin and Tseytlin using the Polyakov path integral [7]. A calculation of the gauge-field beta function has also been carried out in [16] up to terms quadratic in α', with compatible results. In [16], the nonabelian case is also considered.

3. Propagator on the annulus

We now would like to consider the effect of the gauge field at the next order in the string loop expansion. To begin, we shall calculate the propagator on the annulus in the presence of a background gauge field. As before, our result will be exact to zeroth order in derivatives of F. Let us take the string world-sheet to be a flat annulus with inner radius a and outer radius b, as shown in fig. 3. The Neumann function for the annulus satisfying the equations

$$\frac{1}{2\pi\alpha'}\Box G(z,z') = -\delta(z-z'), \tag{3.1}$$

$$\left.\frac{\partial}{\partial r}G(z,z')\right|_{r=b} = -\frac{\alpha'}{b}, \qquad \left.\frac{\partial}{\partial r}G(z,z')\right|_{r=a} = 0 \tag{3.2}$$

is given by the infinite series [17]

$$G(z,z') = -\alpha'\left[\ln|z-z'| + \sum_{n=1}^{\infty}\ln\left(\left|1-\left(\frac{a}{b}\right)^{2n}\frac{b^2}{z\bar{z}'}\right|\left|1-\left(\frac{a}{b}\right)^{2n}\frac{z\bar{z}'}{a^2}\right|\right)\right.$$
$$\left.+\sum_{n=1}^{\infty}\ln\left(\left|1-\left(\frac{a}{b}\right)^{2n}\frac{z}{z'}\right|\left|1-\left(\frac{a}{b}\right)^{2n}\frac{z'}{z}\right|\right)\right]. \tag{3.3}$$

Note that since the annulus has finite area, one cannot impose $\partial G/\partial r = 0$ both at $r=a$ and $r=b$. This is because eq. (3.1) and Gauss's theorem require $\int_{r=a,b}(\partial G/\partial n)\,ds = -2\pi\alpha'$, which is incompatible with $\partial G/\partial r \equiv 0$. The simplest allowable boundary condition is to set the normal derivative equal to a nonzero constant on some boundary component, as in (3.2). This change in the Neumann boundary conditions amounts to shifting the background field $\bar{X}^{\mu}$ by its average value on the boundary.

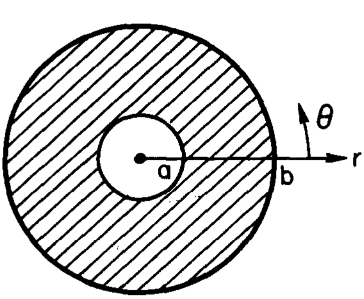

Fig. 3. The world sheet for the one-loop orientable open string diagram may be represented by an annulus. The modulus of the annulus is a/b.

We now would like to find the propagator on the annulus in the presence of the gauge field $F_{\mu\nu}$. The boundary condition is then given by eq. (2.4). However, when $F \neq 0$, this boundary condition together with (3.1) and Gauss's theorem would not permit the propagator to be single-valued. The simplest modification consistent with a single-valued propagator is to require

$$\frac{\partial}{\partial r} G_{\mu\nu}(z, z') - \frac{i}{r} F_{\lambda\mu} \frac{\partial}{\partial \theta} G^{\lambda}{}_{\nu}(z, z') = \begin{cases} 0 & \text{at } r = a \\ \dfrac{-\alpha'}{b} \delta_{\mu\nu} & \text{at } r = b . \end{cases} \tag{3.4}$$

The propagator defined by (3.1) and (3.4) is then

$$G_{\mu\nu}(z, z') = -\tfrac{1}{2}\alpha' \left[\tilde{G}_{\mu\nu}(z, z') + \tilde{G}^{\dagger}_{\mu\nu}(z, z') \right], \tag{3.5}$$

where

$$\begin{aligned} \tilde{G}_{\mu\nu}(z, z') = {} & \delta_{\mu\nu} \ln(z - z') + \delta_{\mu\nu} \sum_{n=1}^{\infty} \ln\left(\left[1 - \left(\frac{a}{b}\right)^{2n} \frac{z'}{z} \right] \left[1 - \left(\frac{a}{b}\right)^{2n} \frac{z}{z'} \right] \right) \\ & + \left(\frac{1 - F}{1 + F} \right)_{\mu\nu} \sum_{n=1}^{\infty} \ln\left(\left[1 - \left(\frac{a}{b}\right)^{2n} \frac{b^2}{z\bar{z}'} \right] \left[1 - \left(\frac{a}{b}\right)^{2n} \frac{z\bar{z}'}{a^2} \right] \right) . \end{aligned} \tag{3.6}$$

When deriving this, it is convenient to diagonalize $F_{\mu\nu}$ in 2×2 blocks,

$$F_{\mu\nu} = \begin{pmatrix} 0 & f_1 & & & \\ -f_1 & 0 & & & \\ & & 0 & f_2 & \\ & & -f_2 & 0 & \\ & & & & \ddots \end{pmatrix}, \tag{3.7}$$

and to adopt a complex basis, so that each of these blocks is replaced by if_n for $n = 1 \dots \frac{1}{2}D$, where D is the dimension of spacetime. Then the propagator becomes a diagonal matrix with $\frac{1}{2}D$ complex entries. Note that the propagator (3.5) can also be written as

$$\begin{aligned} -\frac{1}{\alpha'} G_{\mu\nu}(z, z') = {} & \delta_{\mu\nu} \ln|z - z'| + \delta_{\mu\nu} \sum_{n=1}^{\infty} \ln\left(\left| 1 - \left(\frac{a}{b}\right)^{2n} \frac{z}{z'} \right| \left| 1 - \left(\frac{a}{b}\right)^{2n} \frac{z'}{z} \right| \right) \\ & + \left(\frac{1 + F^2}{1 - F^2} \right)_{\mu\nu} \sum_{n=1}^{\infty} \ln\left(\left| 1 - \left(\frac{a}{b}\right)^{2n} \frac{b^2}{z\bar{z}'} \right| \left| 1 - \left(\frac{a}{b}\right)^{2n} \frac{z\bar{z}'}{a^2} \right| \right) \\ & - \left(\frac{F}{1 - F^2} \right)_{\mu\nu} \sum_{n=1}^{\infty} \ln\left(\frac{\left[1 - \left(\frac{a}{b}\right)^{2n} \frac{b^2}{z\bar{z}'} \right] \left[1 - \left(\frac{a}{b}\right)^{2n} \frac{z\bar{z}'}{a^2} \right]}{\left[1 - \left(\frac{a}{b}\right)^{2n} \frac{b^2}{\bar{z}z'} \right] \left[1 - \left(\frac{a}{b}\right)^{2n} \frac{\bar{z}z'}{a^2} \right]} \right) . \end{aligned} \tag{3.8}$$

From eq. (3.8) it is obvious that this propagator reduces to the usual one (3.3) when $F_{\mu\nu} = 0$.

A remark on the beta function is in order. In sect. 2, we computed the beta function for A_μ on the upper half-plane, i.e. at open string tree level. Since the conformal anomaly should be a local effect, the beta function should not depend on the topology of the world sheet. We may now easily check this for the annulus. Suppose we compute the graph of fig. 1 using the propagator (3.8). In the limit $z \to z'$, we find a logarithmic divergence in $G(z, z')$ due to the first term $\ln|z - z'|$. Moreover, there are only two other terms in the series which can give rise to logarithmic divergences. One of these is the term $\ln|1 - a^2/z\bar{z}'|$ when $z \to z'$ on the inner boundary, and the other is the term $\ln|1 - z\bar{z}'/b^2|$ when $z \to z'$ on the outer boundary. In either case, the propagator loop in fig. 1 gives the divergence

$$G_{\mu\nu}(z \to z') = -2\pi\alpha'\left(1 + \frac{1+F^2}{1-F^2}\right)_{\mu\nu} \ln \Lambda,$$

exactly as in the tree level case (2.12).

4. Vacuum amplitude on the annulus

In this section, we will compute the vacuum-to-vacuum amplitude for the annulus by integrating the stress-energy tensor over the modulus a/b, as outlined in [18]. The stress-energy tensor may be computed using the propagator (3.8) on the annulus in the presence of a gauge field found in the previous section.

The vacuum amplitude or partition function is given by the Polyakov path integral

$$Z = \sum_{\text{topologies}} \int \mathcal{D}g\,\mathcal{D}X\,\mathrm{e}^{-S[g,X]}, \tag{4.1}$$

where

$$S[g, X] = \frac{1}{4\pi\alpha'}\int \mathrm{d}^2z\,\sqrt{g}\,g^{ab}\,\partial_a X^\mu \partial_b X_\mu + \frac{i}{2\pi\alpha'}\int_{r=a,b} \mathrm{d}s\,A_\mu \frac{\partial}{\partial s} X^\mu$$

and g_{ab} is the world-sheet metric, with line element $\mathrm{d}s$. Previously, we have always set $g_{ab} = \delta_{ab}$. The gauge invariance of (4.1) allows us to choose this metric here as well, provided we integrate over the modulus of the annulus. We will take the outer radius b to be one, so the modulus is just the inner radius a. Therefore the partition function can be written as an integral over $\mathrm{d}a$ of

$$Z(a) = \int_{\substack{\text{annulus of}\\ \text{modulus } a}} \mathcal{D}X\,\mathrm{e}^{-S[\delta_{ab}, X]}.$$

The partition function $Z(a)$ for an annulus of modulus a can be computed using the stress-energy tensor as follows. In terms of $Z(a)$, the stress-energy tensor may be written

$$T_{ab} = -\frac{2\pi\alpha'}{\sqrt{g}} \frac{\delta \ln Z(a)}{\delta g^{ab}}.$$

Therefore, we can write

$$\frac{\partial}{\partial a} \ln Z(a) = -\frac{1}{2\pi\alpha'} \int d^2z \sqrt{g} \frac{\delta g^{ab}}{\delta a} T_{ab}. \tag{4.2}$$

Therefore, if we know the stress-energy tensor, we can compute Z up to a multiplicative constant (since the integration constant in (4.2) is undetermined).

The factor $\delta g_{ab}/\delta a$ in (4.2) requires some explanation. Above, we have fixed the gauge in (4.1) by choosing a flat metric on the annulus and integrating over the radius of the inner boundary. Alternately, we could fix the boundaries and integrate over a one-parameter family of "Teichmüller deformations" of the metric [19]. The metric change δg_{ab} induced by a change δa of the inner radius is the Teichmüller deformation of g_{ab} which is conformally equivalent to changing the inner radius by an amount δa while keeping the metric fixed. The following calculation of $\delta g_{ab}/\delta a$ will clarify this procedure.

Let us map the annulus of inner radius a to an annulus of radius $a + \delta a$ via a mapping

$$z \to z' = z + \varepsilon(z, \bar{z}), \tag{4.3}$$

which satisfies the boundary conditions

$$\varepsilon(z, \bar{z}) = \begin{cases} 0 & \text{at } r = 1 \\ e^{i\theta}\delta a & \text{at } r = a, \end{cases} \tag{4.4}$$

so that the inner boundary is displaced a distance δa perpendicular to itself. In complex coordinates, the original flat metric is

$$ds^2 = |dz|^2 = g_{z\bar{z}}\, dz\, d\bar{z} + g_{\bar{z}z}\, d\bar{z}\, dz,$$

with

$$g_{z\bar{z}} = g_{\bar{z}z} = \tfrac{1}{2}, \qquad g^{z\bar{z}} = g^{\bar{z}z} = 2,$$
$$g_{zz} = g_{\bar{z}\bar{z}} = g^{zz} = g^{\bar{z}\bar{z}} = 0. \tag{4.5}$$

The mapping (4.3) induces a metric on the new annulus, which to first order is

$$ds^2 = |dz'|^2 = |1 + \partial_z \varepsilon|^2 |\, dz + \partial_{\bar{z}}\varepsilon\, d\bar{z}|^2. \tag{4.6}$$

The traceless part of the change in the metric,

$$\delta g_{\bar{z}\bar{z}} = \partial_{\bar{z}}\varepsilon, \qquad \delta g_{zz} = \overline{\partial_{\bar{z}}\varepsilon}, \tag{4.7}$$

cannot be removed by a conformal transformation and is potentially a Teichmüller deformation.

In order for (4.7) to be a Teichmüller deformation, it must be in the kernel of $P_1^\dagger$, where P_1 is the operator introduced in [19] to map a diffeomorphism into the traceless part of the metric variation induced by it. This amounts to imposing the condition

$$\partial_z \delta g_{\bar{z}\bar{z}} = \partial_{\bar{z}} \delta g_{zz} = 0, \tag{4.8}$$

which means that $\varepsilon(z, \bar{z})$ must be a harmonic function, i.e.

$$\partial_z \partial_{\bar{z}} \varepsilon = 0. \tag{4.9}$$

Eqs. (4.9) and (4.4) uniquely determine ε to be

$$\varepsilon(z, \bar{z}) = \delta a \frac{a}{\bar{z}} \left(\frac{1 - |z|^2}{1 - a^2} \right). \tag{4.10}$$

Consequently,

$$\delta g_{\bar{z}\bar{z}} = \partial_{\bar{z}}\varepsilon = -\frac{1}{\bar{z}^2} \left(\frac{a \delta a}{1 - a^2} \right),$$

and we find that

$$\frac{\partial \ln Z(a)}{\partial a} = \frac{4a}{1 - a^2} \int \mathrm{d}^2 z \left(\frac{1}{\bar{z}^2} T_{zz} + \frac{1}{z^2} T_{\bar{z}\bar{z}} \right). \tag{4.11}$$

The stress-energy tensor is determined by the two-point function

$$T_{zz} = \tfrac{1}{2} \langle \partial_z X^\mu(z) \partial_{z'} X_\mu(z') \rangle_{z=z'} .$$

Since the limit $z \to z'$ is singular, a regularization is required. Using the propagator (3.5) found in the previous section, we get the regularized stress-energy tensor

$$T_{zz} = \frac{1}{2} \left[\frac{\partial^2 G^\mu{}_\mu(z, z')}{\partial z \, \partial z'} - \frac{\alpha' D}{2(z - z')^2} \right] = \frac{D}{4\pi z^2} \sum_{n=1}^{\infty} \frac{a^{2n}}{(1 - a^{2n})^2}, \tag{4.12}$$

where D is, as usual, the spacetime dimension.

Notice that this result is completely independent of $F_{\mu\nu}$. Substituting T_{zz} and its complex conjugate $T_{\bar{z}\bar{z}}$ into (4.11) and integrating over z gives

$$\frac{\partial \ln Z(a)}{\partial a} = \frac{4aD}{1-a^2} \sum_{n=1}^{\infty} \frac{a^{2n}}{(1-a^{2n})^2} \int_a^1 \frac{\mathrm{d}r}{r^3} = -\frac{2D}{a} \sum_{n=1}^{\infty} \frac{a^{2n}}{(1-a^{2n})^2}. \quad (4.13)$$

Integration over $\mathrm{d}a$ gives, up to a constant [20],

$$\ln Z(a) = -D \sum_{n=1}^{\infty} \ln(1-a^{2n}),$$

and therefore,

$$Z(a) = \prod_{n=1}^{\infty} (1-a^{2n})^{-D}. \quad (4.14)$$

So far, we have neglected the ghost contribution, which is obviously independent of $F_{\mu\nu}$. Introducing it, one recovers the usual partition function for the annulus [21],

$$Z = \int \frac{\mathrm{d}a}{a^3} \prod_{n=1}^{\infty} (1-a^{2n})^{-24}. \quad (4.15)$$

An interesting feature of this result is that it does not depend on F. However, note that the above method would not be sensitive to an overall F-dependent factor. In fact, using the operator formalism, we will find in sect. 6 that the partition function actually has the form

$$Z_{\mathrm{F}} = \det(1+F) Z, \quad (4.16)$$

where Z is the partition function (4.15).

5. Open string spectrum in a background gauge field

In this section we will discuss the spectrum of open strings in a constant background abelian gauge field. In this limit, the oscillator modes, Virasoro algebra, and spectrum may be found exactly. We assume that the electromagnetic field couples to charges q_1 and q_2 at the ends of the string. We will work in Minkowski space and represent the world-sheet as the strip shown in fig. 4, with σ spacelike and τ timelike. The string action in Minkowski space is (setting $\alpha' = \frac{1}{2}$)

$$S = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathrm{d}\tau \int_0^{\pi} \mathrm{d}\sigma \left(\dot{X}_\mu \dot{X}^\mu - X'_\mu X'^\mu \right)$$
$$- \frac{1}{\pi} \int_{-\infty}^{+\infty} \mathrm{d}\tau \left[q_1 A_\mu \dot{X}^\mu(\sigma=0) + q_2 A_\mu \dot{X}^\mu(\sigma=\pi) \right], \quad (5.1)$$

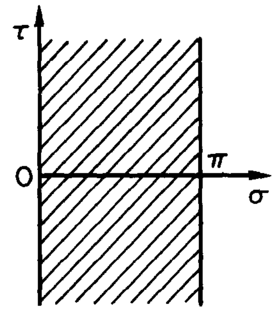

Fig. 4. Strip representation of the open string world sheet.

where $\dot{X}^{\mu} = \partial_{\tau} X^{\mu}$ and $X'_{\mu} = \partial_{\sigma} X_{\mu}$. Varying this action gives the usual wave equation $\ddot{X}_{\mu} - X''_{\mu} = 0$ with the boundary conditions

$$X'_{\mu} = q_1 F_{\mu\nu} \dot{X}^{\nu} \qquad \text{at } \sigma = 0,$$

$$X'_{\mu} = -q_2 F_{\mu\nu} \dot{X}^{\nu} \qquad \text{at } \sigma = \pi. \tag{5.2}$$

We shall take $F_{\mu\nu}$ to have only spatial components. Since F is a real antisymmetric tensor, we can make a rotation to put it into block diagonal form (3.7). The blocks are independent, so we can concentrate on only one of them,

$$\begin{pmatrix} 0 & f \\ -f & 0 \end{pmatrix}.$$

In these two dimensions, it is convenient to introduce complex fields

$$X_+ = \sqrt{\tfrac{1}{2}}\,(X_1 + iX_2), \qquad X_- = \sqrt{\tfrac{1}{2}}\,(X_1 - iX_2) = X_+^{\dagger}\,.$$

(Note that these are *spatial* components, and have nothing to do with the light-cone coordinates, which will never appear in our discussion.) The boundary conditions (5.2) then take the form

$$X'_+(0) = -i\alpha \dot{X}_+(0), \qquad X'_+(\pi) = i\beta \dot{X}_+(\pi), \tag{5.3}$$

where $\alpha = q_1 f$ and $\beta = q_2 f$.

We first consider the case when $q_1 + q_2 \neq 0$, and then discuss the neutral string case. When the net charge is nonzero, the normal mode expression for X_+ is

$$X_+ = x_+ + i\left[\sum_{n=1}^{\infty} a_n \psi_n(\tau, \sigma) - \sum_{m=0}^{\infty} b_m^{\dagger} \psi_{-m}(\tau, \sigma)\right]. \tag{5.4}$$

Here the ψ_n are the normalized mode functions that satisfy the wave equation and the boundary conditions (5.3),

$$\psi_n = |n-\varepsilon|^{-1/2}\cos[(n-\varepsilon)\sigma+\gamma]e^{-i(n-\varepsilon)\tau}, \tag{5.5}$$

where n is an integer, and $\varepsilon=(1/\pi)(\gamma+\gamma')$ with $\gamma=\tan^{-1}\alpha$ and $\gamma'=\tan^{-1}\beta$. There are some important differences between (5.4) and the mode expansion for the ordinary string [22]. Note that there is no term in the expansion (5.4) linear in τ. This is because no function linear in τ and σ can satisfy the boundary conditions when $\alpha+\beta\neq 0$. Also, since the integrals of the mode functions ψ_n are nonzero, x_+ is not identified with a center-of-mass coordinate.

The mode functions satisfy the orthogonality relation

$$\int_0^\pi \frac{d\sigma}{\pi}\overline{\psi_n}(\tau,\sigma)\left[i\overleftrightarrow{\partial}_\tau+\alpha\delta(\sigma)+\beta\delta(\pi-\sigma)\right]\psi_m(\tau,\sigma)=\delta_{mn}\,\mathrm{sgn}(n-\varepsilon). \tag{5.6}$$

where $\phi\overleftrightarrow{\partial}_\tau\psi\equiv\phi\partial_\tau\psi-\psi\partial_\tau\phi$. Also, each ψ_n is orthogonal in this inner product to a constant mode, i.e.

$$\int_0^\pi \frac{d\sigma}{\pi}\left[i\overleftrightarrow{\partial}_\tau+\alpha\delta(\sigma)+\beta\delta(\pi-\sigma)\right]\psi_n(\tau,\sigma)=0. \tag{5.7}$$

Moreover, these modes, together with the constant mode, are complete in the sense that for any function $\phi(\tau,\sigma)$, the following relation holds:

$$\begin{aligned}\phi(\tau,\sigma) = &\sum_{n=-\infty}^{\infty}\mathrm{sgn}(n-\varepsilon)\psi_n(\tau,\sigma)\\ &\times\int_0^\pi\frac{d\tilde\sigma}{\pi}\overline{\psi_n}(\tau,\tilde\sigma)\left[i\overleftrightarrow{\partial}_\tau+\alpha\delta(\tilde\sigma)+\beta\delta(\pi-\tilde\sigma)\right]\phi(\tau,\tilde\sigma)\\ &+\frac{1}{\alpha+\beta}\int_0^\pi d\tilde\sigma\left[i\overleftrightarrow{\partial}_\tau+\alpha\delta(\tilde\sigma)+\beta\delta(\pi-\tilde\sigma)\right]\phi(\tau,\tilde\sigma).\end{aligned}$$

The canonical momentum $P_+(\tau,\sigma)$ is defined in the usual way,

$$P_+(\tau,\sigma)=\frac{\partial\mathscr{L}}{\partial\dot X_+}=\frac{1}{\pi}\dot X_-(\tau,\sigma)-\frac{1}{\pi}A_-(\tau,\sigma)[q_1\delta(\sigma)+q_2\delta(\pi-\sigma)]. \tag{5.8}$$

Because of the gauge interaction terms in (5.1), the canonical momentum contains a boundary contribution. If we choose the gauge such that

$$A=\tfrac{1}{2}f(X_1\hat X_2-X_2\hat X_1),$$

we get

$$\pi P_+ = \dot{X}_- + \tfrac{1}{2} i X_- [\alpha\delta(\sigma) + \beta\delta(\pi - \sigma)] . \tag{5.9}$$

Although the boundary contribution in (5.9) is gauge dependent, we shall see later that physical quantities are gauge invariant.

We can use the orthogonality relations (5.6) and (5.7) to invert the Fourier expansion (5.4) and write the Fourier coefficients x_+, a_n and b_n in terms of X_+, $\dot{X}_+$, X_-, and $\dot{X}_-$. We can then use eq. (5.9) and its complex conjugate to find $\dot{X}_\pm$ in terms of $P_\mp$ and $X_\pm$. The result is

$$a_n = \int_0^\pi \mathrm{d}\sigma \overline{\psi_n}(\tau, \sigma) \left[P_-(\tau, \sigma) - \frac{i}{\pi} [(n - \varepsilon) + \tfrac{1}{2}\alpha\delta(\sigma) + \tfrac{1}{2}\beta\delta(\pi - \sigma)] X_+(\tau, \sigma) \right],$$

$$b_m = \int_0^\pi \mathrm{d}\sigma \psi_{-m}(\tau, \sigma) \left[P_+(\tau, \sigma) - \frac{i}{\pi} [(m + \varepsilon) - \tfrac{1}{2}\alpha\delta(\sigma) - \tfrac{1}{2}\beta\delta(\pi - \sigma)] X_-(\tau, \sigma) \right],$$

$$x_+ = \frac{1}{\alpha + \beta} \int_0^\pi \mathrm{d}\sigma \left[\pi i P_-(\tau, \sigma) + [\tfrac{1}{2}\alpha\delta(\sigma) + \tfrac{1}{2}\beta\delta(\pi - \sigma)] X_+(\tau, \sigma) \right] .$$

Using the canonical commutation relations of the theory,

$$[X_\mu(\tau, \sigma), X_\nu(\tau, \sigma')] = 0, \qquad [P_\mu(\tau, \sigma), P_\nu(\tau, \sigma')] = 0,$$
$$[X_\mu(\tau, \sigma), P_\nu(\tau, \sigma')] = i\delta_{\mu\nu}\delta(\sigma - \sigma') , \tag{5.10}$$

for $\mu, \nu = \pm$, we find that the Fourier modes satisfy

$$[a_n, a_m^\dagger] = \delta_{nm}, \qquad [a_n, a_m] = [a_n^\dagger, a_m^\dagger] = 0,$$
$$[b_n, b_m^\dagger] = \delta_{nm}, \qquad [b_n, b_m] = [b_n^\dagger, b_m^\dagger] = 0, \tag{5.11}$$

and all of the a oscillators commute with all of the b oscillators. We also find the zero-mode commutator

$$[x_+, x_-] = \frac{\pi}{\alpha + \beta} , \tag{5.12}$$

where $x_- = x_+^\dagger$. The commutation relation (5.12) may seem surprising, since in ordinary string theories, the constant modes x_μ commute. In fact, eq. (5.12) implies that $(i/\pi)(\alpha + \beta)x_-$ behaves as a conjugate momentum operator for x_+. It is easily verified that eqs. (5.11) and (5.12) do not change if one makes a gauge transformation $\boldsymbol{A} \to \boldsymbol{A} + \nabla\Lambda$. Therefore, the spectrum will be gauge invariant.

Next we study the Virasoro algebra of the theory. The Virasoro operators L_n are defined as usual [22] to be the Fourier components of the constraint equation, i.e.

$$\tfrac{1}{2}\sum_{\mu=0}^{D-1} :\left(\dot{X}_\mu \pm X_\mu'\right)^2: = \frac{1}{\pi}\sum_{n=-\infty}^{\infty} L_n e^{in(\tau\pm\sigma)}. \tag{5.13}$$

We can find the contribution of X_1 and X_2 to eq. (5.13) by using (5.4) and (5.5) together with the fact that

$$\tfrac{1}{2}\sum_{\mu=1}^{2}\left(\dot{X}_\mu \pm X_\mu'\right)^2 = \left(\dot{X}_+ \pm X_+'\right)\left(\dot{X}_- \pm X_-'\right).$$

This gives for $n \geqslant 0$,

$$\begin{aligned} L_n^{(1,2)} = &\sum_{m=1}^{\infty}\left[(m-\varepsilon)(n+m-\varepsilon)\right]^{1/2} a_m^\dagger a_{n+m} \\ &+ \sum_{m=0}^{\infty}\left[(m+\varepsilon)(n+m+\varepsilon)\right]^{1/2} b_m^\dagger b_{n+m} \\ &+ \sum_{m=0}^{n-1}\left[(m+\varepsilon)(n-m-\varepsilon)\right]^{1/2} b_m a_{n-m} \end{aligned} \tag{5.14}$$

and

$$L_{-n}^{(1,2)} = L_n^{(1,2)},$$

where the superscript (1,2) refers to the contribution of these two spatial components of X^μ, to which must be added the contributions of the other twenty-four components.

It is a straightforward but tedious exercise to calculate the commutators of the Virasoro operators. The result is

$$\left[L_n^{(1,2)}, L_m^{(1,2)}\right] = (n-m)L_{n+m}^{(1,2)} + \delta_{n+m,0}\left[\tfrac{2}{12}(n^3-n) + n\varepsilon(1-\varepsilon)\right]. \tag{5.15}$$

This is the expected result, apart from the extra c-number piece $n\varepsilon(1-\varepsilon)\delta_{n+m,0}$ which is linear in n, and hence can be eliminated by a shift $L_0 \to L_0 + \frac{1}{2}\varepsilon(1-\varepsilon)$. This corresponds to changing the normal ordering constant α_0 from its original value of 1 to $1-\frac{1}{2}\varepsilon(1-\varepsilon)$.

We can now describe the spectrum of the theory. The effect of the magnetic field is to shift the frequencies of the a oscillators by $-\varepsilon$ and those of the b oscillators by $+\varepsilon$. The zero mode operators have been drastically changed: the string total momentum operators $p_\pm$ do not appear in the mode expansion, while instead we get extra Fourier operators $b_0^\dagger$ and b_0 which create and annihilate quanta of frequency ε. In ordinary string theory, we would take our states to be eigenstates of $p_\pm$, which commute with L_0. We would not take them to be eigenstates of the center-of-mass operators, since these do not commute with L_0. However, when $F \neq 0$, x_+ and x_- do commute with L_0, since the zero modes are missing from the mode expansion (5.14). Therefore, the states may be taken to be eigenstates of x_+, for example.

Now L_0 does not depend on $x_\pm$, so there is an infinite degeneracy if the first two dimensions are not compact. (The degeneracy is finite if these dimensions are compact. This case is discussed in appendix B.) This situation is the same as that of a charged particle moving in a constant magnetic field: the states form equally spaced Landau levels of infinite degeneracy, the separation between consecutive levels being proportional to qB. In our case, $q = q_1 + q_2$ and $B = f$, and the operators b_0 and $b_0^\dagger$ move the string from one Landau level to another. The frequency separation ε is proportional to qB when qB is sufficiently small.

The excited states of the string are created by acting on the ground-state wavefunctions with the oscillator creation operators. At the first excited level we have the states $a_1^\dagger|x_+\rangle$ which have $(\text{mass})^2 = -\frac{1}{2}\varepsilon(1+\varepsilon)$ and hence are tachyonic, and the states $b_1^\dagger|x_+\rangle$ which have $(\text{mass})^2 = \frac{1}{2}\varepsilon(3-\varepsilon)$ and hence are massive. This is reminiscent of the behavior of Yang-Mills fields in the presence of a constant chromomagnetic field condensate, where one gluonic polarization becomes tachyonic and the other becomes massive [23].

We have so far considered the case where the string has nonzero total charge. When the total charge vanishes, several changes occur. First, since $\varepsilon = 0$, the string has integer modes, as it does in the absence of $F_{\mu\nu}$ (although the phase shift γ does not vanish). More importantly, the zero modes change completely, as can be seen from eq. (5.12), which is not well-defined when $\alpha + \beta \to 0$. The $m = 0$ mode in (5.4) also disappears. On the other hand, we now have the freedom to add to X_+ a term proportional to $\tau - i\alpha\sigma$, which satisfies the boundary conditions when $\alpha + \beta = 0$. Now we can write the mode expansion as

$$X_+(\tau,\sigma) = \frac{x_+ + p_-\left[\tau - i\alpha\left(\sigma - \frac{1}{2}\pi\right)\right]}{\sqrt{1+\alpha^2}} + i\sum_{n=1}^{\infty}\left[a_n\psi_n(\tau,\sigma) - b_n^\dagger\psi_{-n}(\tau,\sigma)\right], \tag{5.16}$$

where ψ_n is given by (5.5) with $\varepsilon = 0$.

The operator p_- which now appears in the mode expansion becomes the minus component of the total momentum when F vanishes. The commutation relations for the zero mode operators are

$$[x_\mu, x_\nu] = 0, \qquad [p_\mu, p_\nu] = 0, \qquad [x_\mu, p_\nu] = i\delta_{\mu\nu}, \tag{5.17}$$

where $\mu, \nu = \pm$. The factor $(1+\alpha^2)^{-1}$ in front of the linear terms in (5.16) was introduced to allow x_μ and p_ν to satisfy the standard commutation relation. As in the charged string case, the oscillator mode operators satisfy (5.11). There is, however, no b_0 mode for the neutral string. We conclude that the Fourier modes of a neutral string satisfy the same commutation relations as they would in the absence of an external magnetic field. The Virasoro operators, when written in terms of the oscillators and the new zero modes, also have the same form as in the absence of the field. In fact, using (5.16) we get

$$\dot{X}_+ \pm X'_+ = \mathrm{e}^{\mp i\gamma}\left[p_+ + \sum_{n=1}^{\infty}\sqrt{n}\left(a_n \mathrm{e}^{-in(\tau\pm\sigma)} + b_n^\dagger \mathrm{e}^{in(\tau\pm\sigma)}\right)\right]. \tag{5.18}$$

Using (5.13) we get

$$L_0 = p_+p_- + \sum_{n=1}^{\infty} n\left(a_n^\dagger a_n + b_n^\dagger b_n\right), \tag{5.19}$$

with $p_+p_- = \frac{1}{2}(p_1^2 + p_2^2)$. Hence, the spectrum of a neutral string is not affected by the magnetic field. However, as we shall see in the next section, the vacuum-to-vacuum amplitude is changed by the field.

6. Vacuum amplitude in the operator formalism

In this section we will compute the vacuum-to-vacuum amplitude in the presence of a constant gauge field, using the operator formalism [21]. Specifically, we will compute the vacuum amplitude

$$Z = -\tfrac{1}{2}\mathrm{Tr}\ln(L_0 - \alpha_0) \tag{6.1}$$

on an annulus, where α_0 is the normal ordering constant and the trace refers to a sum over string states and an integration over the loop momentum. In the absence of gauge fields, (6.1) gives the one-loop orientable contribution to the cosmological constant. We shall restrict our attention to the neutral string case, and use the results of the previous section to compute Z in the presence of a background magnetic field.

We have learned that when the string is neutral, both the normal ordering constant and the Virasoro operators are the same as in the absence of the gauge field. The naive conclusion seems to be that Z should not depend on $F_{\mu\nu}$. However, the momentum integration in (6.1) requires some care. The momentum integration measure in (6.1) is determined by putting the system in a box of length L. From (5.16) we see that x_μ and p_μ are rescaled by a factor of $(1+\alpha^2)^{-1/2}$ with respect to X_μ. If X_μ goes from 0 to L, then x_μ goes from 0 to $L\sqrt{1+\alpha^2}$. Therefore the momentum eigenvalues are $p_n = (2\pi n/L)(1+\alpha^2)^{-1/2}$. This means that, in the thermodynamic limit, the sum over momentum becomes

$$\sum_n \to \frac{L}{2\pi}\sqrt{1+\alpha^2}\int \mathrm{d}p\,.$$

Hence the integration over $\mathrm{d}p_1$ and $\mathrm{d}p_2$ introduces a factor $1+\alpha^2 = 1+f^2$ (setting $q_1 = -q_2 = 1$), and similarly for each of the other pairs of transverse momentum. The vacuum amplitude Z_{F} in the presence of F will differ from the free amplitude Z given by (4.15) by a factor of

$$\prod_{i=1}^{D/2}\left(1+f_i^2\right) = \det(1+F)\,.$$

This result does not depend on the specific way we rescale x_μ and p_μ to satisfy the commutation relations (5.17). All of the F-dependence could be put instead in p_μ. Then instead of (5.16) we would have

$$X_+ = x_+ + \frac{p_-}{1+\alpha^2}\left[\tau - i\alpha\left(\sigma - \tfrac{1}{2}\pi\right)\right] + i\sum_{n=1}^{\infty}\left[a_n\psi_n - b_n^\dagger\psi_{-n}\right] \tag{6.2}$$

and again (5.17) is satisfied. Now the momentum integration in (6.1) is the usual $\int \mathrm{d}^Dp/(2\pi)^D$. However, L_0 now depends on F. Instead of eqs. (5.18) and (5.19), we get

$$\dot{X}_+ \pm X_+' = \frac{\mathrm{e}^{\mp i\gamma}}{\sqrt{1+\alpha^2}}\left[P_+ + \sum_{n=1}^{\infty}\sqrt{n}\left(a_n\mathrm{e}^{-in(\tau\pm\sigma)} + b_n^\dagger\mathrm{e}^{in(\tau\pm\sigma)}\right)\right]$$

and

$$L_0 = \frac{p_+p_-}{\sqrt{1+\alpha^2}} + \sum_{n=1}^{\infty} n\left(a_n^\dagger a_n + b_n^\dagger b_n\right),$$

respectively. Therefore the same overall factor $\det(1+F)$ emerges when we perform the momentum integration, via the jacobian of the transformation $p_\mu \to p_\mu(1+\alpha^2)^{-1/2}$.

In conclusion, we find that in the case of the neutral string, the F-dependence of the vacuum amplitude may be written

$$Z_{\mathrm{F}} = \det(1+F)Z\,. \tag{6.3}$$

Note that this result agrees with the result of [7], up to finite terms. We believe that the result of [7] corresponds instead to the case of equal, rather than opposite, charges on the boundaries, and that a path integral computation of Z_F in the case of the neutral string would agree with (6.3). This has been checked by an independent calculation which will be discussed in more detail in a subsequent paper [24].

It is tempting to interpret the cosmological constant Z_F on the annulus as a string loop correction to the effective action [7]. However, Z_F is plagued with infinities, since Z has the usual infinities associated with the tadpole divergence [21]. Actually, at the one-loop level, we cannot really talk of open strings alone, since closed string poles automatically appear in the scattering amplitudes associated with the annulus. The inclusion of closed string modes may be needed to obtain a finite or renormalizable theory, and a more complete approach should include closed string backgrounds, such as the metric, from the start. We shall pursue this in a later paper [24].

7. Conclusions

In this paper, we have developed methods for doing exact calculations of open string physics in a slowly-varying background field-strength. In this approximation, we have done the generic string tree and one-loop calculations. The string tree calculation was used to determine the beta function, and hence the equation of motion, for the background gauge field, as well as an effective action from which this equation of motion can be derived. By a straightforward extension of these techniques, it would be possible to obtain corrections arising from the inclusion of higher derivatives of the background gauge field strength.

The string-loop calculation, aside from being interesting in its own right, should be thought of as a technical warmup for the problem of finding the gauge field corrections to the equations of motion for the *closed* string background fields (in particular, the metric). The point is that the partition function on the annulus has background gauge field divergences coming from the integration over the modular parameter. If we take the attitude of Fischler and Susskind [25] and renormalize them by appropriate world-sheet sigma model counterterms, we will obtain gauge-field dependent string loop contributions to the beta functions of the closed string background fields. In particular, we should recover the right gauge field contributions to the metric beta function (or, equivalently, the gauge field contribution to the matter energy-momentum tensor). The details of such calculations, which will make heavy use of methods described here, will be reported in a subsequent paper [24].

Appendix A

ALTERNATE DERIVATION OF THE BETA FUNCTION

In this appendix, we derive the beta function for $F_{\mu\nu}$, starting from the free open string propagator (2.8) and using the quadratic and cubic interaction vertices in

(2.5) to form all possible divergent one-loop graphs proportional to $\partial_\tau \bar{X}^\mu$. The results are the same as those found in sect. 2. We set $2\pi\alpha' = 1$ in this appendix.

The Fourier transform of the free propagator in the upper half-plane is [26]

$$G(z, z') = \int \frac{\mathrm{d}p}{2\pi} \frac{\mathrm{e}^{ip(\tau-\tau')}}{2|p|} \left(\mathrm{e}^{-|p||\sigma-\sigma'|} + \mathrm{e}^{-|p|(\sigma+\sigma')}\right).$$

Since the interactions are all at the boundary, we need this propagator only for $\sigma = \sigma' = 0$, i.e.

$$G(\tau, \tau') = \int \frac{\mathrm{d}p}{2\pi} \frac{\mathrm{e}^{ip(\tau-\tau')}}{|p|}. \tag{A.1}$$

Notice that

$$G^{-1}(\tau, \tau') \equiv \partial_\tau \partial_{\tau'} G(\tau, \tau') = \int \frac{\mathrm{d}p}{2\pi} |p| \mathrm{e}^{ip(\tau-\tau')} \tag{A.2}$$

is the inverse boundary propagator. In fact,

$$\int \mathrm{d}\tilde{\tau}\, G^{-1}(\tau, \tilde{\tau}) G(\tilde{\tau}, \tau') = \delta(\tau - \tau'). \tag{A.3}$$

The one-loop divergence is obtained by summing the infinite series of graphs in fig. 5. We are summing all tadpole graphs with any number of insertions of the vertex $\frac{1}{2} i F_{\mu\nu} \xi^\nu \partial_\tau \xi^\mu$. Each graph with n such insertions represents a sum of 2^n terms with all possible orderings of the derivatives ($\xi^\nu \partial_\tau \xi^\mu$ or $(\partial_\tau \xi^\mu)\xi^\nu$ at each vertex). These 2^n terms are all equal, as is easily checked using the antisymmetry of $F_{\mu\nu}$ and integration by parts. We therefore restrict attention to the ordering shown in fig. 5 and include a combinatorial factor 2^n. The contribution from the graph with n insertions is

$$I_n = 2^n \left(\tfrac{1}{2} i\right)^{n+1} \int \mathrm{d}\tau\, \mathrm{d}\tau_1 \ldots \mathrm{d}\tau_n \left(\nabla^\nu F_{\nu_1 \mu} \partial_\tau \bar{X}^\mu\right) F^{\nu_1}{}_{\nu_2} \ldots F^{\nu_n}{}_\nu$$
$$\times G(\tau, \tau_1) \partial_{\tau_1} G(\tau_1, \tau_2) \ldots \partial_{\tau_n} G(\tau_n, \tau).$$

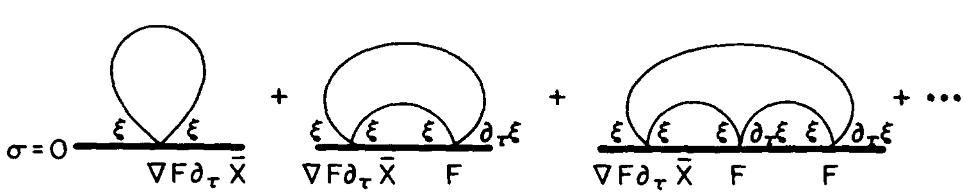

Fig. 5. One-loop graphs for the gauge field beta function.

In the approximation of slowly-varying fields, the background fields $F(\overline{X})$ all can be evaluated at the point τ, as is usual in background field calculations [14]. Then we can integrate over $\tau_2, \dots \tau_n$, using the relation

$$\int d\tau_k \partial_{\tau_{k-1}} G(\tau_{k-1}, \tau_k) \partial_{\tau_k} G(\tau_k, \tau_{k+1}) = -\delta(\tau_{k+1} - \tau_{k-1}),$$

valid for $n-1 \geqslant k \geqslant 2$, which may be derived by integrating by parts and applying (A.2) and (A.3). A similar relation applies to τ_n, with τ replacing the variable τ_{k+1}.

For even n, integrating over $\tau_2, \dots, \tau_n$ just leaves a factor of $(-1)^{n/2}\delta(\tau - \tau_1)$, and therefore the integral reduces to

$$I_n = \tfrac{1}{2} i \int d\tau \left(\nabla^\nu F_{\lambda\mu} \partial_\tau \overline{X}^\mu \right) \left[(iF)^n \right]^\lambda{}_\nu G(\tau, \tau), \tag{A.4}$$

where

$$G(\tau, \tau) = \frac{1}{\pi} \int \frac{dp}{|p|} = -2 \ln \Lambda$$

is logarithmically divergent in the ultraviolet cutoff Λ of sect. 2. For odd n, we find instead

$$I_n = -\tfrac{1}{2} \int d\tau d\tau_1 \left(\nabla^\nu F_{\lambda\mu} \partial_\tau \overline{X}^\mu \right) (F^n)^\lambda{}_\nu G(\tau, \tau_1) \partial_{\tau_1} G(\tau_1, \tau),$$

which is proportional to an odd momentum integral $\int dp p/(2\pi|p|^2)$, and hence vanishes identically. Thus we obtain the total one-loop counterterm

$$\Delta S_I = \sum_{n=0}^{\infty} I_n = \tfrac{1}{2} i \int d\tau \partial_\tau \overline{X}^\mu \nabla^\nu F_\mu{}^\lambda (1 - F^2)^{-1}_{\lambda\nu} G(\tau, \tau). \tag{A.5}$$

Therefore, the beta function is

$$\beta_A^\mu = \nabla^\nu F^{\mu\lambda} (1 - F^2)^{-1}_{\lambda\nu},$$

as found in sect. 2.

Appendix B

STRINGS IN A MAGNETIC FIELD ON A TORUS

We shall consider what happens when x_1 and x_2 are compact, with length L. In this appendix, we will rename x_1 and x_2 to be x and y. Since only the zero modes are important in the analysis below, we will find a close analogy to the situation

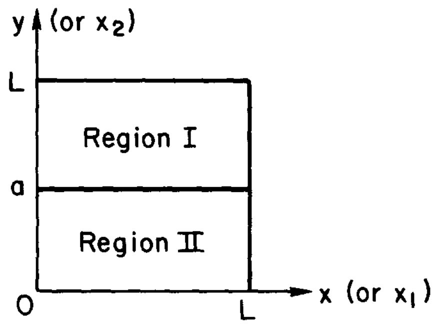

Fig. 6. The torus is represented as a square with opposite sides of length L identified. It is divided into two regions I and II by the circles $y = 0$ and $y = a$.

when a charged particle moves in a magnetic field on a torus. To clarify the analogy, it will be convenient to use the notations $q \equiv q_1 + q_2$, and $B \equiv f$, since these quantities will be analogous to the charge of the particle and the magnetic field, respectively.

A constant magnetic field on a torus is actually a monopole field; no smooth single-valued vector potential $\boldsymbol{A}$ exists in this case. We may separate the torus into two regions I and II (see fig. 6), and place in the two regions smooth potentials that are related by a gauge transformation at their intersection, which consists of the circles $y = 0$ and $y = a$. Specifically, we may choose

$$A_1 = \begin{cases} -By & 0 \leqslant y < a, \\ -B(y-L) & a \leqslant y < L, \end{cases}$$

$$A_2 = 0. \tag{B.1}$$

In real coordinates, the zero-mode operators x and y satisfy the commutation relation

$$[x, y] = \frac{\pi i}{\alpha + \beta} = \frac{\pi i}{qB}, \tag{B.2}$$

so that $k \equiv qBy/\pi$ may be taken to be a conjugate momentum operator for x. States are created by applying the $a_n^\dagger$ and $b_m^\dagger$ to the "momentum" eigenstates $|k\rangle$. A gauge transformation $\boldsymbol{A} \to \boldsymbol{A} - \chi\hat{\boldsymbol{x}}$ has the effect of shifting the zero mode y to $y + \chi$, while x is unchanged. Thus $k \to k + (1/\pi)qB\chi$, and it follows that the zero-mode wave function $\langle x|k\rangle = \mathrm{e}^{ikx}$ is multiplied by a phase factor $\mathrm{e}^{iqB\chi x/\pi}$. In particular, it follows from (B.1) that the transition function from region I to region II is $\mathrm{e}^{iqBLx/\pi}$.

The transition function $e^{iqBLx/\pi}$ is single-valued only if the Dirac quantization condition

$$\frac{1}{\pi}qBL^2 = 2\pi n \tag{B.3}$$

is satisfied for some integer n. Since the torus is not divided in the x-direction, the zero-mode wave function $\langle x|k\rangle = e^{ikx}$ must be single-valued. This implies that the k eigenvalues are quantized, $k_m = 2\pi m/L$. Therefore, using the Dirac quantization condition (B.3), we find that y can take only the values

$$y_m = \frac{m}{n}L. \tag{B.4}$$

Since $y + L$ is the same as y, it follows that there are only n independent wave functions $\langle x|k_m\rangle$, and hence the degeneracy of any state is n. This is as should be expected, since the degeneracy of a Landau level for a particle of charge q in a uniform magnetic field B is also n, with n given by (B.3). We conclude this appendix by demonstrating this fact.

Assume the gauge potential in the xy plane is $\boldsymbol{A} = -By\hat{\boldsymbol{x}}$. If the plane were infinite, the eigenstates of the hamiltonian would have the form

$$\psi_{l,m}(x, y) = e^{-ik_m x}\tilde{\psi}_l(y - y_m), \tag{B.5}$$

where $\tilde{\psi}_l$ is the lth wave function of a linear harmonic oscillator. The energy eigenvalues $E_{l,k} = qB(l + \frac{1}{2})$, are independent of k. However, on the torus of fig. 6, the wave functions must be single-valued as functions of y, and should therefore be related in the two regions by the appropriate gauge transformations on the $y = 0$ and $y = a$ circles. We shall try to satisfy this condition by taking linear combinations of the infinite-plane wave functions with the same value of l, but different values of k, so they will still be eigenstates of the hamiltonian.

We begin by writing a wave function that is properly matched on the boundary of the regions,

$$\psi_l(x, y) = \begin{cases} \sum_{m=-\infty}^{\infty} c_m e^{-ik_m x}\tilde{\psi}_l(y - y_m), & 0 \leqslant y < b \\ e^{iqBLx/\pi} \sum_{m=-\infty}^{\infty} c_m e^{-ik_m x}\tilde{\psi}_l(y - y_m), & a \leqslant y < L. \end{cases} \tag{B.6}$$

We now impose the single-valuedness conditions

$$\psi_l(x,0) = \psi_l(x, L), \qquad \psi_l(0, y) = \psi_l(L, y). \tag{B.7}$$

The first of these gives $k_m = 2\pi m/L$ for some integer m. Substituting (B.6) into the second condition gives

$$e^{iqBLx/\pi}\sum_{m=-\infty}^{\infty} c_m e^{-2\pi imx/L}\tilde{\psi}_l\left(L-\frac{2\pi^2 m}{qBL}\right)=\sum_{m=-\infty}^{\infty} c_m e^{-2\pi imx/L}\tilde{\psi}_l\left(\frac{-2\pi^2 m}{qBL}\right). \tag{B.8}$$

The Dirac quantization condition (B.3) can be written as $qBL = 2\pi^2 n/L$, and hence eq. (B.8) can be rewritten

$$\sum_{m=-\infty}^{\infty} c_m e^{2\pi i(n-m)x/L}\tilde{\psi}_l\left(\left(1-\frac{m}{n}\right)L\right)=\sum_{m=-\infty}^{\infty} e^{-2\pi imx/L}\tilde{\psi}_l\left(-\frac{m}{n}L\right). \tag{B.9}$$

Now we substitute $m \to m+n$ in the l.h.s. of (B.9) to obtain the result

$$\sum_{m=-\infty}^{\infty} c_{m+n} e^{-2\pi imx/L}\tilde{\psi}_l\left(-\frac{m}{n}L\right)=\sum_{m=-\infty}^{\infty} c_m e^{-2\pi imx/L}\tilde{\psi}_l\left(-\frac{m}{n}L\right)$$

from which it follows that $c_{m+n} = c_m$. There are precisely n linearly independent solutions of this last condition, and hence n independent wave functions at the lth Landau level. We may choose them to be

$$\psi_{l,j}(x,y) = \text{const}\sum_{m=-\infty}^{\infty} e^{-2\pi i(j+mn)x/L}\tilde{\psi}_l(y-y_{j+mn}),$$

where $j = 0, 1, \ldots, n-1$.

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