Physics 221

Department of Physics
The Citadel

Lecture Notes

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Angular Momentum and Torque
In 3 Dimensions

Announcements

- Homework Set 12: due Wednesday
  Sections 11.5 and 12.1 – 12.3.
  Problems: Two on torque & angular momentum vectors, plus Ch. 12: 9, 11, 13, 14, 45, 59.

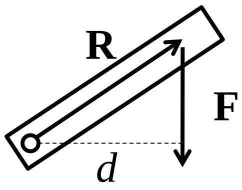
Today: 3 dimensional torque and angular momentum, precession, gyroscopes.

Torque and Angular Momentum

Torque: $\boldsymbol{\tau} = \mathbf{R} \times \mathbf{F}$

$\tau = Fd$

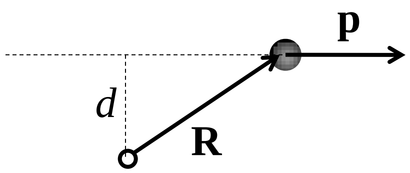
away from you



Angular momentum: $\mathbf{L} = \mathbf{R} \times \mathbf{p}$

$L = pd = mvd$

away from you



Torque and Angular Momentum

Newton's Law: $\mathbf{F} = d\mathbf{p}/dt$,

Rotational Analog: $\boldsymbol{\tau} = d\mathbf{L}/dt$.

Newton's Law holds in an inertial frame.

$\boldsymbol{\tau} = d\mathbf{L}/dt$ holds in an inertial frame, or about the CM regardless of acceleration.

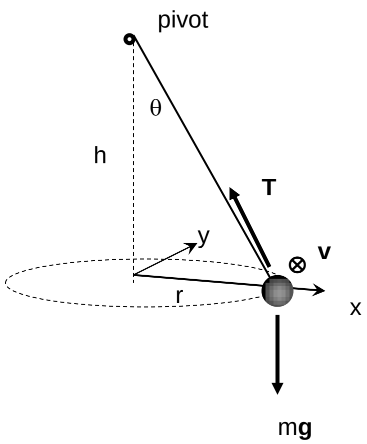
This allows the rotational motion of a rigid body about the CM to be calculated independently of its CM motion.

Conical Pendulum

A ball circles on the end of a string. The speed is v and the circle has radius r.

What is v, given r, h?

We've done this before…



Centripetal acceleration is due to tension:

$mv^2/r = T \sin\theta$.

Forces balance in y direction:

$T \cos\theta = mg$.

$mv^2/r = mg \tan\theta = mg\, r/h$.

$v^2 = r^2 g / h$.

Conical Pendulum

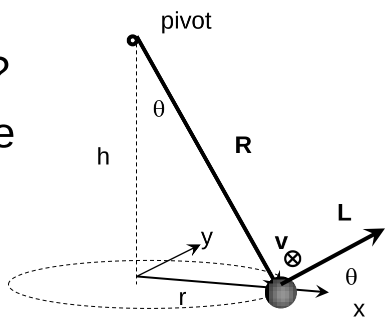
Is the angular momentum about the pivot constant?

What is $\mathbf{L} = m\mathbf{R}\times\mathbf{v}$ when the ball is on the $+x$ axis?

$\mathbf{R}$ and $\mathbf{v}$ are perpendicular, so $L = mRv$.

Direction?

$\mathbf{L}$ is perpendicular to $\mathbf{R}$ and $\mathbf{v}$: use right-hand-rule.



Conical Pendulum

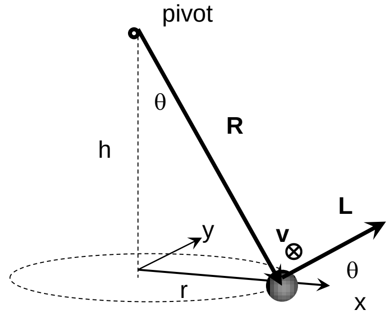
Components:

$L_x = Rmv \cos\theta = hmv$

$L_y = 0$

$L_z = Rmv \sin\theta$

$= rmv$



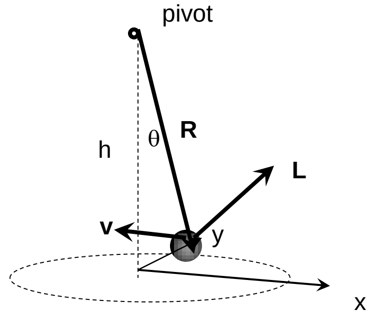
What happens to **L** as the ball rotates? Where does it point when the ball is on the $+y$ axis?

Conical Pendulum

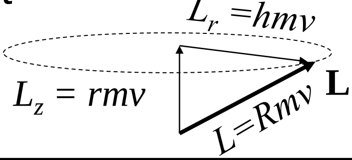
$L_x = 0$

$L_y = hmv$,

$L_z = rmv$ doesn't change.



L makes a circle of radius L_r = hmv (I'll call that the radial component) about the z axis.



Conical Pendulum

Note that only

$L_z = rmv$

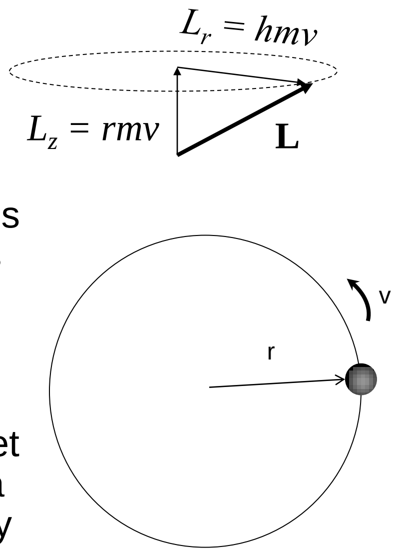
remains constant.

You could have calculated this component using fixed axis methods in 2 dimensions.

$I = mr^2$, $\omega = v/r$,

$L_z = I\omega = mvr$.

You can always use this to get the component of L along a rotation axis, but not for any other component.



Conical Pendulum

What is the total external torque on the system about the pivot?

$\boldsymbol{\tau} = \mathbf{R} \times m\mathbf{g}$

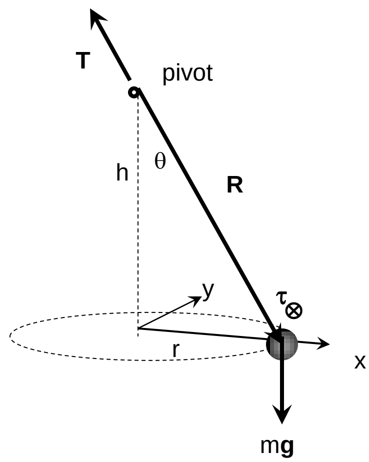
Magnitude:

force times lever arm ,

$\tau = mgr$

Direction:

The same as $\mathbf{v}$ – into the page in the position shown.

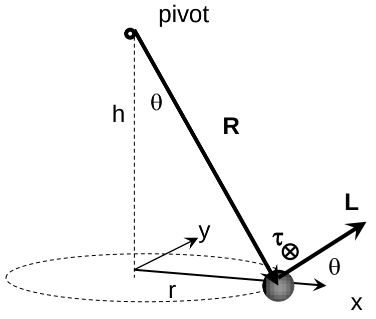


Conical Pendulum

$\boldsymbol{\tau} = mgr\,\hat{\mathbf{v}} = d\mathbf{L}/dt$

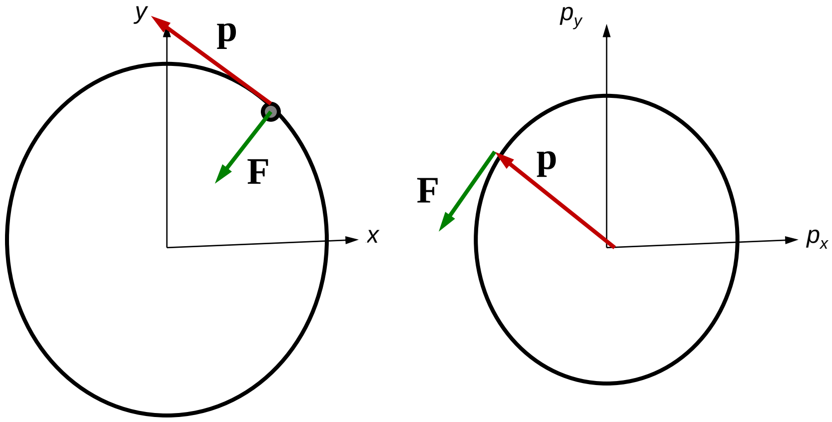
$\boldsymbol{\tau}$ is perpendicular to **L**, so the magnitude of **L** doesn't change, only the direction.

This is similar to uniform circular motion, where the direction of the momentum changes but not the magnitude.



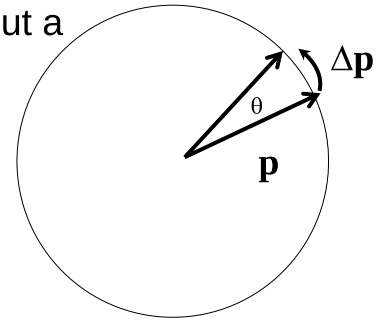
Uniform Circular Motion

The velocity and momentum vectors rotate at the same rate as the position vector.



Uniform Circular Motion

In uniform circular motion, the momentum vector traces out a circle as the particle goes around. The force is perpendicular to the momentum, so



$\Delta\mathbf{p} = \mathbf{F}\,\Delta t$ is perp. to $\mathbf{p}$.

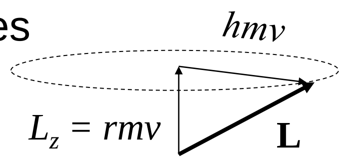
$|\Delta\mathbf{p}| = p\,\theta = p\,\omega\,\Delta t$.

$F = \Delta p/\Delta t = p\omega$.

More familiar: $F = mv\omega = mv^2/r$.

Conical Pendulum

Similarly, the vector $\mathbf{L}$ rotates in a circle with radius



$L_r = hmv$, so that

$$dL/dt = hmv\omega$$

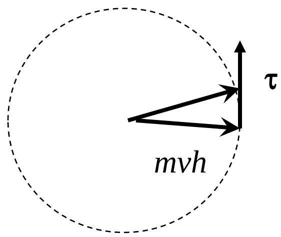
$$= hmv^2/r$$

$\tau = mgr = dL/dt$

implies $mgr = hmv^2/r$

Then $v^2 = gr^2/h$. Good!

That's what we got before.

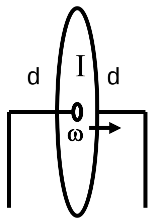


top view of angular momentum and torque

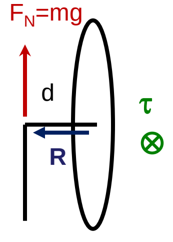
Gyroscope

A gyroscope spins rapidly about the axis shown. The axis is supported at both ends.

If the support at the right is removed, what happens to the gyroscope?



Gyroscope



Find the torque about the CM.

$\tau = mgd$ into the page.

What is L?

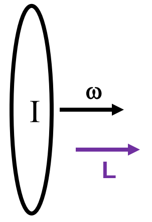
$L = I\omega$ to the right.

Does the magnitude of L change?

- No.

Does the direction change?

- Yes.



Gyroscope

What is the period?

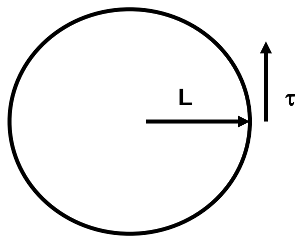
L traces a circle of radius $2\pi L$ in time T.

Rate $dL/dt = 2\pi L/T = \tau$

$T = 2\pi L/\tau$

$= 2\pi I\omega/(mgd)$.

This is the **precession** period of the gyroscope, since **L** follows the axis.



Supplement: Tipped Wheel

We demonstrated a tipped wheel in class, but didn't Calculate the motion. The following optional supplement shows how this could be done.

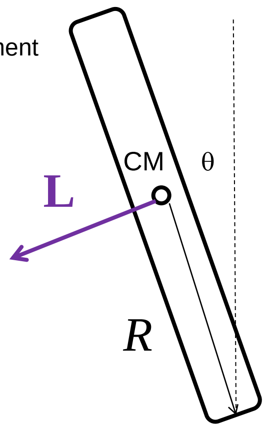
Consider a bicycle wheel of radius R and mass m rolling away from you at speed v tilted at angle θ.

$L = I\omega$ along the axle.

If all the mass is on the rim (heavy tire with light spokes and axle),

$I = mR^2$ and

$L = mR^2\omega = mRv$



Rolling away at speed v.

Supplement: Tipped Wheel

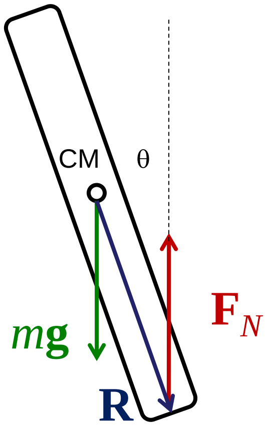
What is the torque on the wheel due to the forces shown?

$\tau = F_N R \sin\theta$

$= mgR \sin\theta$

Direction: toward you.

Angular momentum changes: $\Delta L = \tau \Delta t$

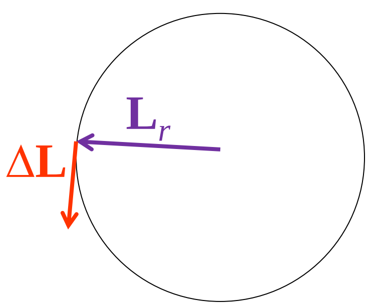


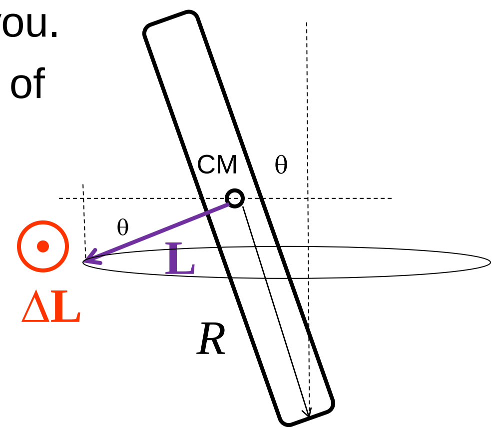
Supplement: Tipped Wheel

$\Delta L = \tau \Delta t$ is toward you.

L rotates on a cone of radius $L_r = L \cos\theta$

View from above:





Rolling away.

Supplement: Tipped Wheel

Using $\tau = mgR \sin\theta$ gives

$\Delta L = \tau\Delta t = mgR \sin\theta \, \Delta t.$

The angular momentum vector makes one complete revolution when the tip goes around the circumference of the cone:

$\Delta L = 2\pi L_r = 2\pi L \cos\theta = 2\pi mRv \cos\theta.$

Period:

$T = \Delta L/\tau = 2\pi mRv \cot\theta / mgR.$

$= 2\pi (v/g) \cot\theta.$

Supplement: Tipped Wheel

The wheel goes in a circle of circumference vT in one period. The radius of this circle is

$r = vT/2\pi$ with $T = 2\pi (v/g) \cot\theta.$

Then the wheel rolls around a circle of radius

$r = (v^2/g) \cot\theta$ or $\tan\theta = v^2/rg.$

But… this analysis cannot be complete!

To go in a circle requires a force to produce the centripetal acceleration. We neglected it on a first pass because we didn’t know yet how the wheel would respond. But now that we know, we should go back and fix the analysis.

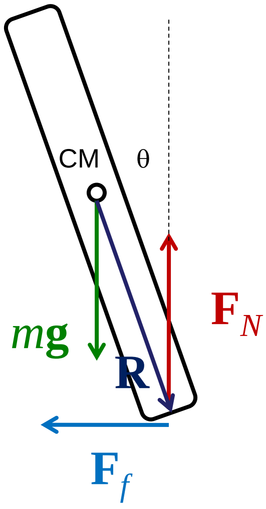
Tipped Wheel: Corrected

We need to add a static frictional force directed inward.

$\tau \;= F_N R \sin\theta - F_f\, R \cos\theta$

$\;= mgR \sin\theta - (mv^2/r_c)\, R \cos\theta$

since the centripetal acceleration is due to the friction. Here, r_c is the radius at which the CM travels as the wheel goes in a circle.



Tipped Wheel: Corrected

The next step is the same as before, with the corrected torque. For one period T,

$\Delta L = \tau T = mR(g \sin\theta - v^2 \cos\theta / r_c)\; T.$

$\quad = 2\pi\, L_r = 2\pi\, L \cos\theta = 2\pi\, mRv \cos\theta.$

Then

$\quad (g \sin\theta - v^2 \cos\theta / r_c)\; T = 2\pi v \cos\theta$

Approximating r_c by r (the distance the bottom of the wheel travels), which is good for small tilt angles, we can use the fact that $2\pi r = vT$ to substitute $T/r_c \approx 2\pi/v$.

Tipped Wheel: Corrected

Then

$(g \sin\theta - v^2 \cos\theta / r_c)\ T = 2\pi v \cos\theta$ with $vT/r_c = 2\pi$:

$gT \sin\theta - 2\pi v \cos\theta = 2\pi v \cos\theta$

$T = 4\pi\,(v/g) \cot\theta$.

This is a big correction: the period is twice what we found by neglecting the horizontal frictional force.

This means the actual radius is in which the wheel moves is also twice as big as originally calculated: $r = (2v^2/g) \cot\theta$, or $\tan\theta = 2v^2/rg$.