Physics 221

Department of Physics
The Citadel

Lecture Notes

S. Yost
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Angular Momentum and Torque
In 3 Dimensions

Announcements

- Homework Set 11: due Wednesday
  Sections 10.9 and 11.1 – 11.3.
  Problems: Ch. 10: 52, 58
  Ch. 11: 3, 16, 28, 32, 33, 39
  Section 11.4 will be included with Ch. 12.

Today we will discuss the vector definition of angular quantities, which is needed in the general case where the axis is not fixed.

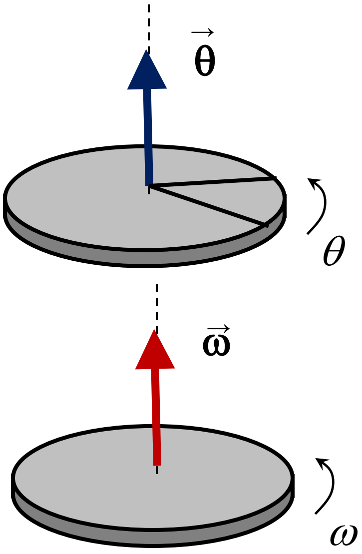
Rotational Direction

- In general rotational motion, the direction of the axis can change.
- We need to generalize the rotational quantities by giving them a magnitude and direction.
- What direction can we give to a rotation?
- An angle can be clockwise or counterclockwise **about some axis**.
- **Take the direction of the axis as the direction of the rotation.**

Rotational Direction

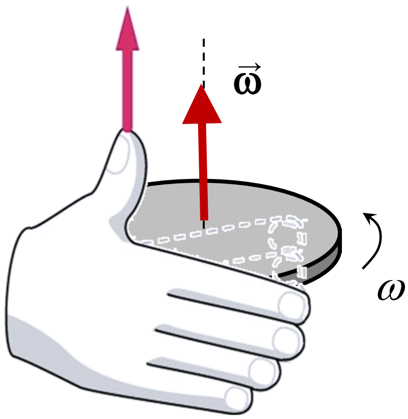
- If the axis is fixed in some arbitary direction, we can define a counterclockwise rotation about the axis to be a vector along that axis.
- If the axis itself is changing, this only gives an instantaneous direction to the rotation.
- That is enough to define the angular velocity vector

$$\omega = d\theta/dt.$$



Angular velocity vector

- If you curl your fingers in the direction of rotation, your thumb points in the direction of the angular velocity.
- This is the first example of a **right-hand rule**. We will see many of these by the end of next semester.



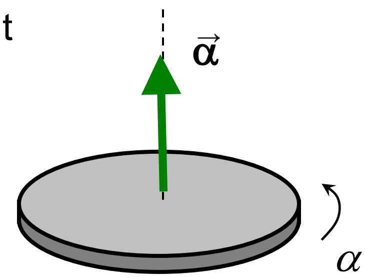
Angular acceleration and torque

- Angular acceleration $\boldsymbol{\alpha} = d\boldsymbol{\omega}/dt$ also points along the instantaneous axis.
- Torque is a vector too, but it isn't necessarily along the axis, so

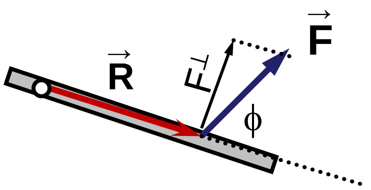
  $\boldsymbol{\tau} = I\boldsymbol{\alpha}$

  only **sometimes** holds for general rotational motion.

  (More on this Friday.)



Torque



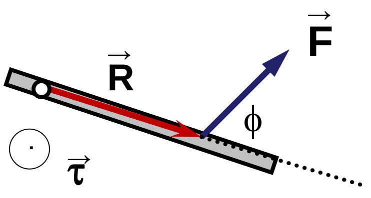
The torque is defined to be the perpendicular component of the force times the distance from the pivot to where it acts:

$$\tau = R\,F^{\perp} = R\,F\sin\phi$$

What defines the direction of the torque vector?

There is only one line perpendicular to the plane of **R** and **F**. The torque is defined to be along that line.

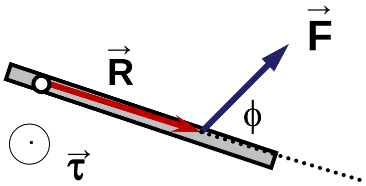
Torque vector



To make torque a vector, we give it a direction perpendicular to both **R** and **F**. This means it points out of the screen.

A vector pointing out of the page is often shown as a circle with a dot in the middle, representing the front of an arrow. The back of the arrow is shown a circle with a cross: the tail-feathers.

Torque vector

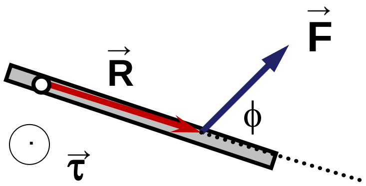


Why out of the page and not in?

The rotation would be counter-clockwise about this axle.

The direction is given by a right-hand rule: Put your fingers along $\vec{\mathbf{R}}$ and curl them toward $\vec{\mathbf{F}}$. The torque points in the direction of your thumb.

Torque vector

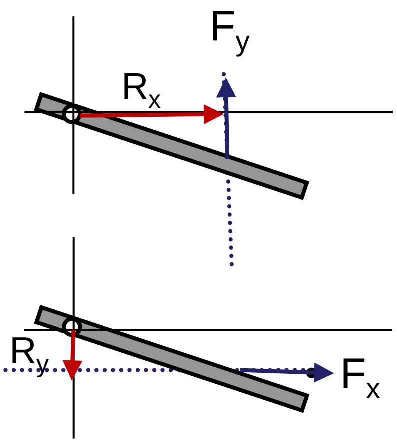


Now we say that the torque points along the *z* axis in this case:

$\tau_x = \tau_y = 0$
$\tau_z = RF \sin \phi$

The torque along any axis is positive when it causes counterclockwise rotation about that axis.

Torque Vector in Components

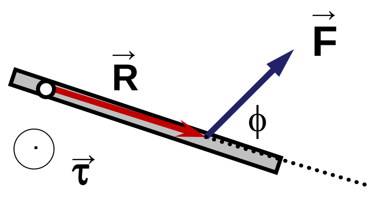


We can find τ_z using cartesian coordinates for **R** and **F**, treating it as the sum of two torques, for F_y and F_x.

F_y gives a counterclockwise torque $+R_xF_y$.
F_x also gives a ccw torque, but it is $-R_yF_x$.
The signs always come out This way for any orientation Of the stick, giving

$$\tau_z = R_xF_y - R_yF_x.$$

Torque Vector in Components



Equations for the torque can be found for an axis in the x or y directions by exactly the same method... The analogs of

$$\tau_z = R_x F_y - R_y F_x$$

are

$$\tau_x = R_y F_z - R_z F_y, \qquad \tau_y = R_z F_x - R_x F_z$$

Don't worry about remembering these equations – there is a mnemonic…

Torque Vector in Components

- When calculating torque, start with $\vec{\mathbf{R}}$ first, then $\vec{\mathbf{F}}$, as when using the right hand rule to get the sign.
- When writing the first term, keep the indices *xyz* in order, **cyclically**: when you get to *z*, go back to *x* and start over.
- Then subtract the term with the indices reversed.

$$\tau_x = R_y F_z - R_z F_y$$

$$\tau_y = R_z F_x - R_x F_z$$

$$\tau_z = R_x F_y - R_y F_x$$

Torque vector

Summary:

$\tau = RF \sin \theta$

$\vec{\tau}$ is perpendicular to $\vec{\mathbf{R}}$ and $\vec{\mathbf{F}}$.

The direction is given by the right-hand rule, curling from $\vec{\mathbf{R}}$ to $\vec{\mathbf{F}}$ and following your thumb to find $\vec{\tau}$.

In components,

$$\tau_x = R_y F_z - R_z F_y$$
$$\tau_y = R_z F_x - R_x F_z$$
$$\tau_z = R_x F_y - R_y F_x$$

This defines a product of vectors $\vec{\mathbf{R}}$ and $\vec{\mathbf{F}}$ called the cross product:

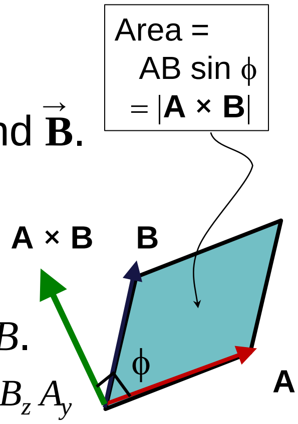
$$\vec{\tau} = \vec{\mathbf{R}} \times \vec{\mathbf{F}}$$

Cross Product

In general, the cross product
$\vec{\mathbf{A}} \times \vec{\mathbf{B}}$ magnitude $AB \sin \phi$.
$\vec{\mathbf{A}} \times \vec{\mathbf{B}}$ is perpendicular to $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$.

The sign is given by the right-hand rule.

$AB \sin \phi$ is also the area of a parallelogram with sides A, B.



Components: $(\mathbf{A} \times \mathbf{B})_x = A_y B_z - B_z A_y$
$(\mathbf{A} \times \mathbf{B})_y = A_z B_x - B_x A_z$
$(\mathbf{A} \times \mathbf{B})_z = A_x B_y - B_y A_x$

Example

Find the torque due to a force
$\mathbf{F} = (3\mathbf{i} - 4\,\mathbf{j} + 2\mathbf{k})\mathrm{N}$ acting at a point
$\mathbf{R} = (2\mathbf{i} + 7\mathbf{j} - 5\mathbf{k})\mathrm{m}$ relative to the pivot point.

$\tau_x = R_y F_z - R_z F_y = (7\times2 - 5\times4)\mathrm{Nm} = -6\ \mathrm{Nm}$
$\tau_y = R_z F_x - R_x F_z = (-5\times3 - 2\times2)\mathrm{Nm} = -19\ \mathrm{Nm}$
$\tau_z = R_x F_y - R_y F_x = (-2\times4 - 7\times3)\mathrm{Nm} = -29\ \mathrm{Nm}$

What is the magnitude of the torque?
$\tau = \sqrt{6^2 + 19^2 + 29^2}\ \mathrm{Nm} = 35.2\ \mathrm{Nm}$

Algebraic Properties

$\mathbf{A} \times \mathbf{B} = -\,\mathbf{B} \times \mathbf{A}$ (anticommutative)

$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) \neq (\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$ (nonassociative)

$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) + \mathbf{B} \times (\mathbf{C} \times \mathbf{A}) + \mathbf{C} \times (\mathbf{A} \times \mathbf{B}) = 0$

(Jacobi identity)

$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}\,(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}\,(\mathbf{A}\cdot\mathbf{B})$

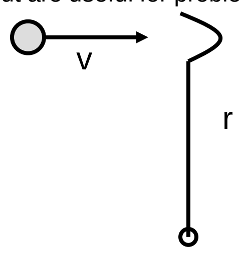
(Lagrange's formula)

$|\mathbf{A} \times \mathbf{B}|^2 + |\mathbf{A}\cdot\mathbf{B}|^2 = A^2B^2\,(\sin^2\phi + \cos^2\phi) = A^2B^2$

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$ (determinant form)

Rigid Body Collision

The next three slides were used only at 8AM, but are useful for problem 8.

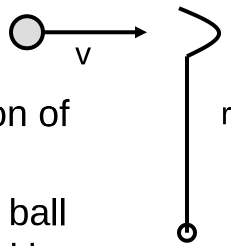


Angular momentum is conserved when a ball is caught, as shown. At the moment of the catch, the ball has angular momentum

$L = I\omega = mr^2\omega = mrv$.

If the catcher is massless, there is no torque on the ball – it continues at speed v, but is now rotating about the center.

Rigid Body Collision



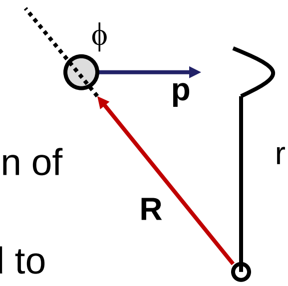
How can we define L so it is conserved for the entire motion of the ball ?

There was no torque before the ball was caught either, so L should be the same when the ball is moving in a straight line:

$L = mrv = rp$ with p the momentum.

even though there is no "rotation" in the usual sense about the axis.

Rigid Body Collision



Note that $L = R\, p \sin \phi$

This is similar to the definition of torque.

A cross product can be used to define angular momentum:

$$\mathbf{L} = \mathbf{R} \times \mathbf{p}$$

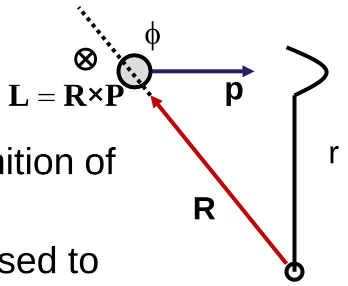
$$d\mathbf{L}/dt = \mathbf{v} \times \mathbf{p} + \mathbf{R} \times d\mathbf{p}/dt$$

$$= m\,\mathbf{v} \times \mathbf{v} + \mathbf{R} \times \mathbf{F} = \boldsymbol{\tau}.$$

zero

Always true: $\boldsymbol{\tau} = d\mathbf{L}/dt.$

Rigid Body Collision



Note that $L = R\,p \sin\phi$

This is similar to the definition of torque.

A cross product can be used to define angular momentum:

$$\mathbf{L} = \mathbf{R} \times \mathbf{p}$$

Direction: Into the page.

After the ball is caught, the direction is clockwise, so this is consistent.