Physics 221

Department of Physics
The Citadel

Lecture Notes

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Rigid Body Rotations

Part 2: Torque

Announcements

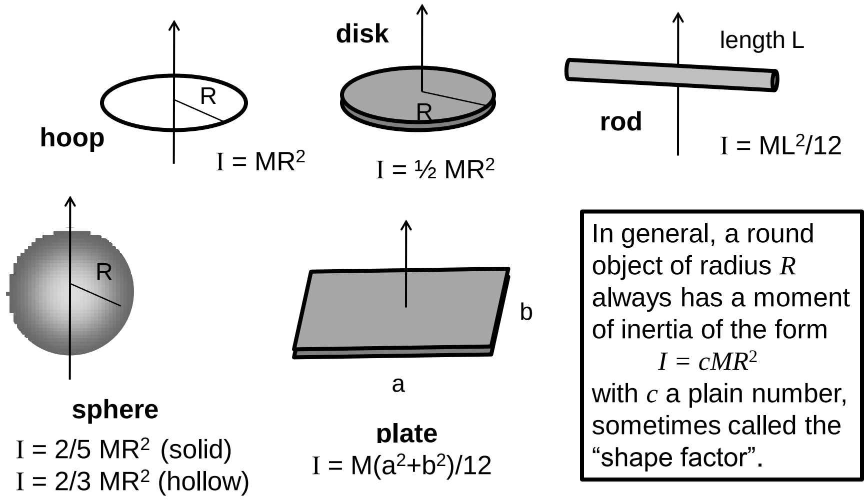
- Homework Set 10: due Wednesday
  Problems: 4, 16, 21, 25, 30, 33, 34, 37 from sections 10.1 – 10.8
- Today: Ch. 10.6 – 10.8
  - moments of inertia, torque

Next: Ch. 10.9 (rolling), Ch. 11: sec. 1 – 4 (angular momentum)

But we will discuss angular momentum for fixed axis rotation first, unlike the text, which does the general case first.

Common Moments of Inertia

All are about the center of mass for uniform objects.



In general, a round object of radius R always has a moment of inertia of the form

$$I = cMR^2$$

with c a plain number, sometimes called the "shape factor".

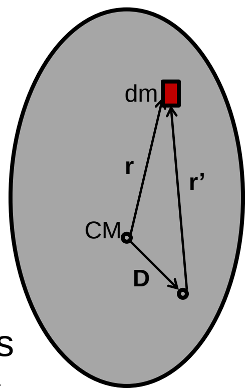
Parallel Axis Theorem

If you know the moment of inertia about an axis through the CM and need it about a parallel axis a distance D away, you can use the parallel axis theorem:

$$I = I_{\mathrm{CM}} + mD^2.$$

Proof: Let the displacement from the CM to the new axis be $\mathbf{D}$.

If the position of dm relative to the CM is $\mathbf{r}$, the position relative to the new axis is $\mathbf{r'} = \mathbf{r} - \mathbf{D}$.



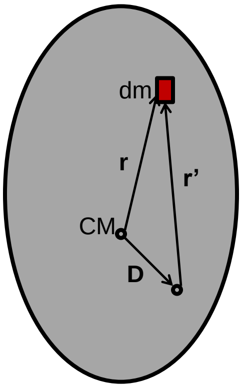
Parallel Axis Theorem

Proof:

$$I = \int r'^2 dm = \int (\mathbf{r} - \mathbf{D})^2 dm$$

$$= \int r^2 dm + D^2 \int dm - 2\mathbf{D} \cdot \int \mathbf{r} dm$$

$$= I_{\mathrm{CM}} + mD^2 - 2\mathbf{D} \cdot \int \mathbf{r} dm$$



The CM is at $\mathbf{r} = 0$, so

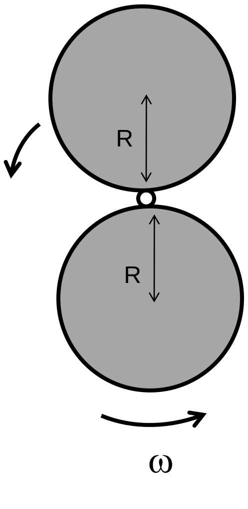
$$0 = \mathbf{r}_{CM} = \frac{1}{m} \int \mathbf{r} dm$$

And $I = I_{\mathrm{CM}} + mD^2$.

Example

A uniform disk of radius 13 cm is pivoted at a point on its edge starts at rest and swings downward.

What is its angular speed about the pivot at the bottom?



Example

This problem can be solved using energy methods:

Initially, $K_i = 0$. At the bottom, $K_f = \frac{1}{2} I \omega^2$.

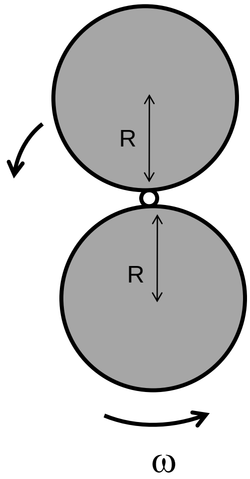
I is the moment of inertia of the disk about the pivot on its edge.

The parallel axis theorem gives

$I = \frac{1}{2} mR^2 + mR^2 = 1.5\, mR^2$

Energy conservation:

$K_f = U_i - U_f$.



Example

What is the PE of the disk, an extended object?

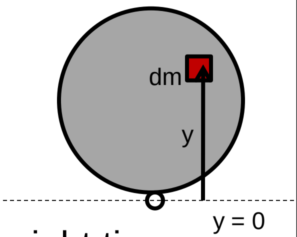
A little piece of the object has PE $dU = gydm$.

$$U = \int dU = \int gydm$$

This can be expressed in terms of

The center of mass:

$$y_{cm} = \frac{\int ydm}{m}, \qquad U = \int gydm = mgy_{cm}$$



The PE of an extended object is its weight times the height of its center of mass.

Example

$U_i = mgR, \quad U_f = -mgR.$

$K_f = ½\, I\, \omega^2 = ¾\, mR^2\, \omega^2.$

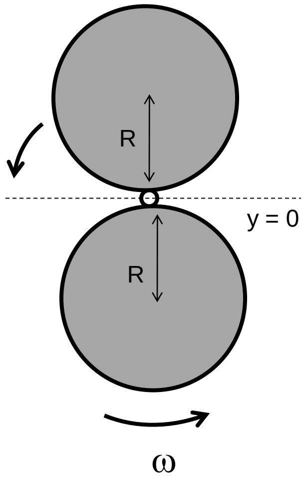
$\quad = U_i - U_f = 2mgR.$

$\omega^2 = (8/3)\, g/R$

$\quad = 2.67\ (9.8\ \text{m/s}^2)(0.13\ \text{m})$

$\quad = 3.40\ \text{s}^{-2}$

$\omega = 1.84$ rad/s

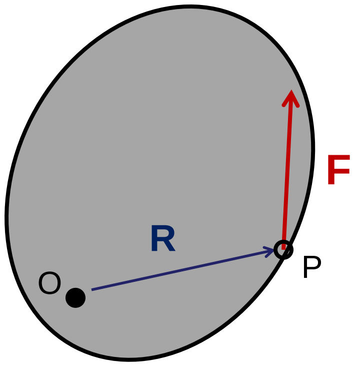


Force on Rigid Body

Suppose a force **F** is applied to a rigid body at a point *P* denoted by the vector **R**, where **R** is measured from a pivot point *O*.

How can we find the angular acceleration of the object about the pivot *O*?

We would like a rotational analog of $F = ma$ for linear motion.



$a \rightarrow \alpha$

$m \rightarrow I$

$F \rightarrow ?$

Torque

The rotational analog of force is called **torque**:

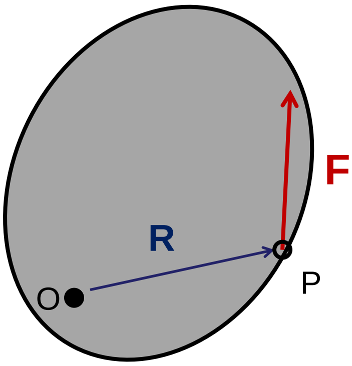
Applying a torque τ to an object produces angular acceleration according to

$$\tau = I\alpha$$

Units: $\mathrm{kg\ m^2\ s^{-2} = N\ m}$.

How is the torque related to the force that causes it?

Does torque have the expected relation to work and power?



$$a \to \alpha$$
$$m \to I$$
$$F \to \tau$$

Torque and Work

First, let us check that this rotational analog of force makes sense in terms of work and power.

What should the rotational analog of $W = F\Delta x$ be?

$$W = \tau\,\Delta\theta$$

Proof: Assume constant α and use the constant acceleration relation $\omega_f^2 - \omega_i^2 = 2\alpha\Delta\theta$:

$$W = \Delta K = \tfrac{1}{2} I\,(\omega_f^2 - \omega_i^2) = I\alpha\,\Delta\theta = \tau\,\Delta\theta.$$

Torque and Power

It is also easy to check the angular analog of the power equation, $P = Fv$. What should it be?

$$P = \tau\omega.$$

Proof: The power is

$$P = dK/dt = d/dt\,(\tfrac{1}{2} I\, \omega^2)$$

$$= I\, \omega\, d\omega/dt = I\, \omega\, \alpha = \tau\, \omega.$$

This rotational analogy seems consistent, but how is torque actually related to the force producing it? We can make the connection through work.

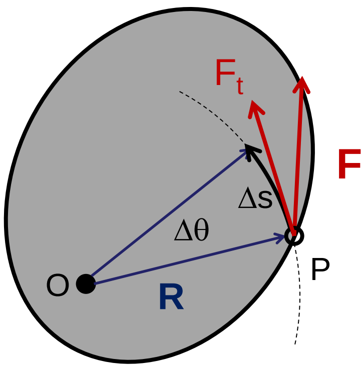
Torque and Force

Let the tangential component of the force along the arc through which the point P moves be denoted F_t.

Then the work done as the object turns through angle $\Delta\theta$ is

$$W = F_t\, \Delta s = F_t\, R\Delta\theta = \tau\, \Delta\theta.$$

Therefore, $\tau = F_t\, R$.



Torque and Force

Another way to express this is that torque is

$$\tau = F_{\perp} R$$

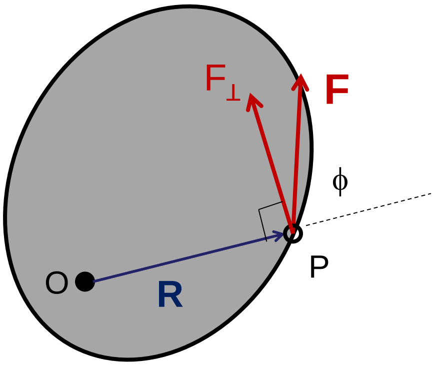
Where R is the distance from O to P and

$F_{\perp} = F \sin \phi$

is the component of F perpendicular to **R**.

Or,

$$\tau = RF \sin \phi.$$



Torque and Force

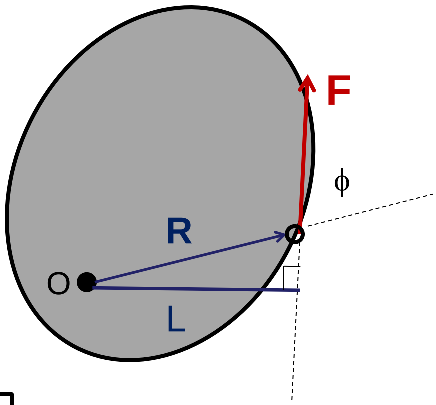
$\tau = RF \sin \phi$ can also be thought of as

$$\tau = F\,(R \sin \phi) = FL,$$

where $L = R \sin \phi$ is called the **lever arm** of the force.

L is the distance from the pivot point to the **line** of the force.

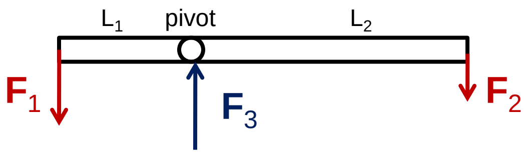
Torque = force × lever arm



Example

Torques can have two directions about a fixed axis: counterclockwise (positive) or clockwise (negative).

$\tau_1 = L_1 F_1$
$\tau_2 = -L_2 F_2$
$\tau_3 = 0$



A force acting at the pivot point produces no torque.

In particular, if an object has $\alpha = 0$, the sum of all torques must balance.

Torque and Force

Summary: torque = force × lever arm = $RF \sin \phi$

All of the rotational analogs expected hold for torque:

$F = ma$	$\tau = I\alpha$
$W = Fx$	$W = \tau\theta$
$P = Fv$	$P = \tau\omega$

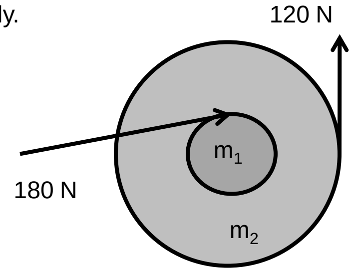
If several forces act on an object at different places, then the net torque determines the angular acceleration.

Example

Used in 8AM class only.

A wheel of radius 24 cm and mass 5.0 kg is attached to an axle of radius 6.5 cm and mass 1.2 kg. Forces are applied as shown.

Find the net torque and angular acceleration.



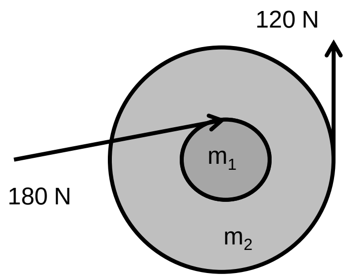
Example

τ = 120N (0.24 m)
– 180 N (0.065 m)
= 17.1 Nm

$I = ½\, m_1 r_1^2 + ½\, m_2 r_2^2$
= 0.1465 kg m^2.

$\alpha = \tau / I$ = 117 rad/s^2.

Direction: ccw.



Inside: r_1 = 6.5 cm
m_1 = 1.2 kg
Outside: r_2 = 24 cm
m_2 = 5.0 kg