Physics 221

Department of Physics
The Citadel

Lecture Notes

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Rotational Motion

Part 1

Announcements

- Homework Set 9: Up later today, due Friday (due to internet problems).

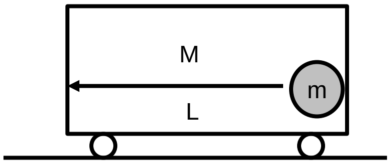
  #8, 16, 18, 20, 26, 29, 37, 43
- Exam 2 is still being graded – back Friday (next problem discussion day)
- Today: Start Chapter 10.
- A new set will be posted tomorrow. It will be due next Wednesday.

Balls in a Freight Car

Cannon balls of total mass m are stacked at the front end of a freight car of length L and mass M.

If the balls are moved to the back end of the car, how far does the car move?

Neglect friction between the car and its track.



Balls in a Freight Car

No external force implies fixed CM position. In general:

$(M+m)x_{cm} = mx_b + Mx_c$.

Take the rear of the car to be initially at $x = 0$.

Initially, $x_b = L$, car CM $x_b = x_0$

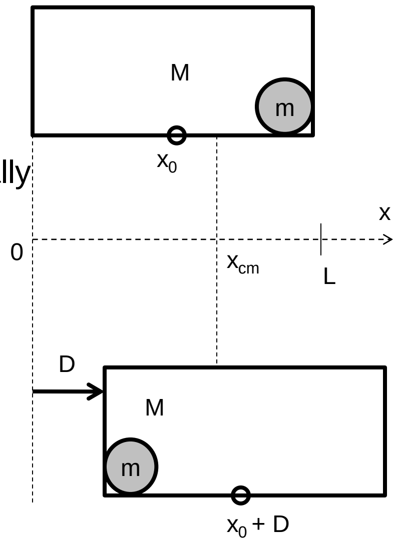
$(M+m)x_{cm} = mL + Mx_0$

After moving the balls:

$x_c = x_0 + D, \quad x_b = D$.

Finally,

$(M+m)x_{cm} = mD + M(x_0 + D)$.



Balls in a Freight Car

Equate the initial and final expressions for the CM:

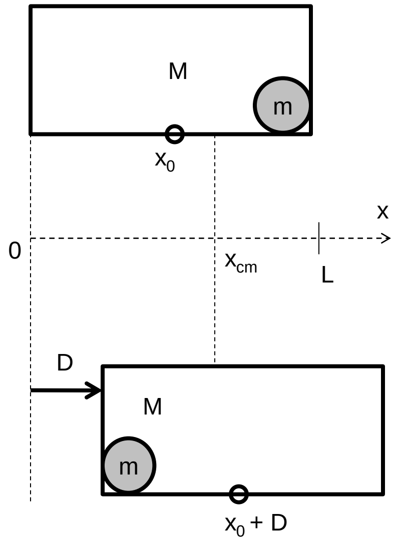
$$mL + Mx_0 = mD + M(x_0 + D).$$

The Mx_0 terms cancel, giving

$$mL = (M+m)D.$$

The car moves a distance

$$D = \frac{mL}{M+m}$$

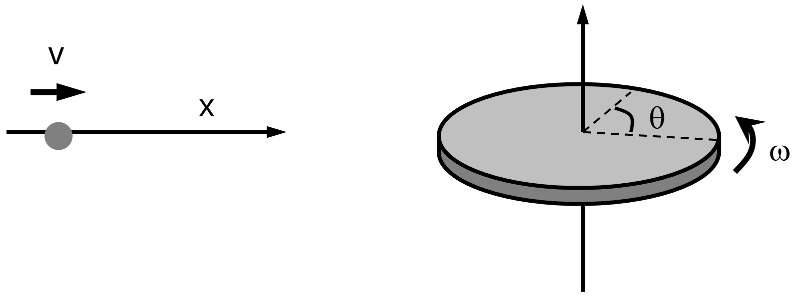


Rotational Motion

- In chapter 10, we consider the rotational motion of rigid bodies.
- This can include rotations about a fixed point, rolling, or some other combination of motion plus rotating.
- We will, however, keep the axis direction fixed until chapter 11. Thus, the rotational part of the motion is effectively one-dimensional.

Rotational Analogies

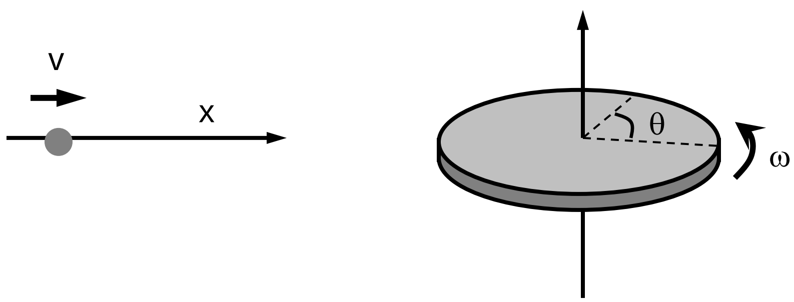
The physics of rotations of a rigid body about a fixed axis is in many ways analogous to the one-dimensional motion of a point particle.



Rotational Analogies

A particle travels a distance x at velocity $v = dx/dt$.

A disk rotates an angle θ at angular velocity $\omega = d\theta/dt$.



Rotational Analogies

Linear Motion:

distance x

velocity $v = dx/dt$

acceleration
$a = dv/dt$

Rotational Motion:

angle θ

angular velocity $\omega = d\theta/dt$

angular acceleration
$\alpha = d\omega/dt$

Constant Acceleration

Linear Motion:

$v = v_0 + at$

$x = x_0 + v_0 t + \frac{1}{2} at^2$

$v^2 = v_0^2 + 2a(x - x_0)$

Rotational Motion:

$\omega = \omega_0 + \alpha t$

$\theta = \theta_0 + \omega_0 t + \frac{1}{2} \alpha t^2$

$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$

Other Useful Definitions

The period is the time it takes to rotate once:

$$T = 2\pi/\omega.$$

The frequency is the number of revolutions per unit time. If an object rotates f times per second, the angle changes by $2\pi f$ per second, so

$$\omega = 2\pi f.$$

Combining these, we see that $f = 1/T$.

Example

A centrifuge accelerates from rest to 20,000 rpm in 5.0 min. What is its average angular acceleration?

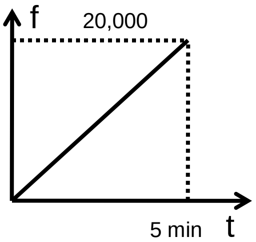
$$\alpha = \omega / t. \qquad t = 300 \text{ s}.$$

$$\omega = 2\pi f = 2\pi\,(20{,}000 \text{ rev} / 60 \text{ s}) = 2094 \text{ rad/s}$$

$$\alpha = (2094 \text{ rad/s}) / 300 \text{ s} = 6.98 \text{ rad/s}^2$$

Example

How many times does the centrifuge spin during these 5 minutes, assuming constant α?



Revolutions $= \theta/2\pi = \omega_{avg}\, t\,/2\pi$

$= f_{avg}\, t$

with

$f_{avg}\, t = \frac{1}{2} f_{max}\, t$

$= \frac{1}{2}$ (20,000 rpm)(5 min)

$=$ 50,000 rev.

Or, use $\theta = \frac{1}{2}\, \alpha\, t^2$ to get the same result with a little more computational effort.

Rotational Dynamics

The cause of acceleration is force:

$$F = ma$$

What is the cause of angular acceleration?

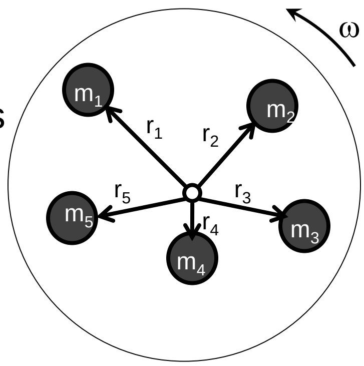
We need a rotational analog of Newton's law. But it is easier to start with rotational energy and make the connection through the work-energy theorem.

Rotational Energy

Consider a massless rotating disk with small masses m_i inserted in it. The total kinetic energy is $K = \Sigma \frac{1}{2} m_i v_i^2$.

It would be nice to express this in terms of the angular velocity of the entire object.

$v = r\omega$ for an object a distance r from the axis.



Rotational Energy

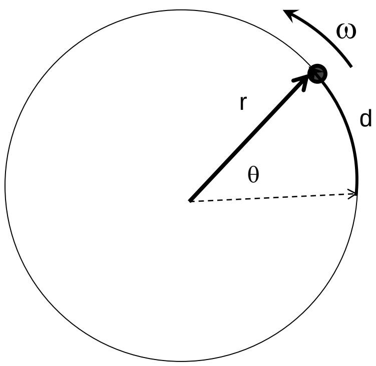
In radians, the distance traveled along an arc of radius r is $d = r\theta$.

The speed is then

$$v = r\, d\theta/dt = r\,\omega.$$

This is an important relation between linear velocity and angular velocity for a rotating object.

Masses further from the axis travel faster.



Rotational KE

- In terms of the angular velocity,

  $K = \Sigma \, ½ \, m_i v_i^2 = ½ \, \Sigma \, m_i r_i^2 \omega^2$

  $= ½$ [mass analog][speed analog]^2.

  $= ½$ [rotational inertia] ω^2.

So $K = ½ \, I \, \omega^2$ with I representing rotational inertia. It is called the moment of inertia.

$I = \Sigma \, m_i r_i^2$ where r is the radial distance of m from the axis.

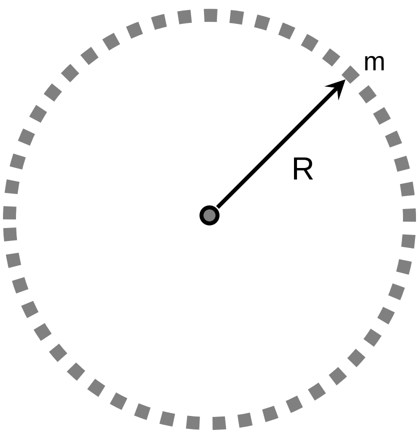
Units: $[I] = \mathrm{kg\ m}^2$.

Moment of Inertia

If there are N masses m a distance R from the axis, then

$$I = \Sigma \, m_i R_i^2 = R^2 \, \Sigma \, m_i = M R^2$$

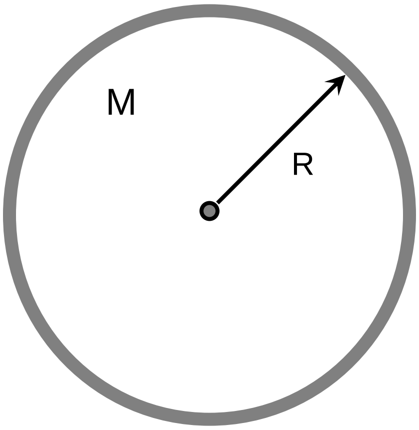
where M is the total mass.



Moment of Inertia

This applies even if the mass distribution is continuous.

For a ring rotating about its center, all the mass is a distance R from the axis, so $I = MR^2$.

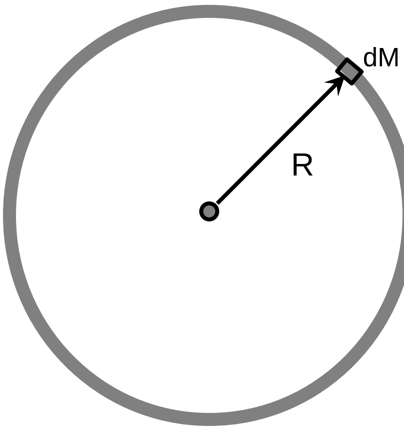


Moment of Inertia

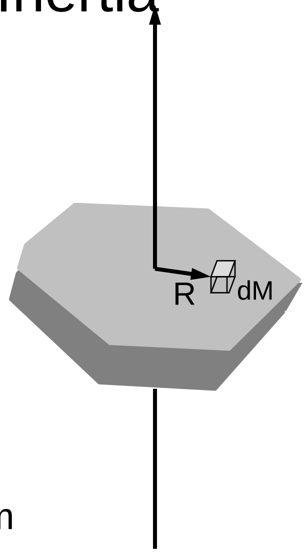
You can think of this as an integral, broken up into a lot of little parts. This is a simple case, giving

$$I = \int R^2\, dM$$

$$= R^2 \int dM = MR^2$$



Moment of Inertia



For a more complicated object, add up contributions

$$dI = R^2\, dM$$

for all little bits of mass dM making up the body. This gives

$$I = \int R^2\, dM$$

R is always the **perpendicular** distance from the mass bit to the axis.

Moment of Inertia of Stick

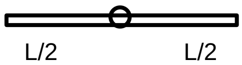
Mass per unit length: λ = M/L.

I about $x = 0$: $\quad dm = \lambda dx$

$$I = \int x^2 dm = \int x^2 \lambda dx = \lambda \int_0^L x^2 dx = \frac{\lambda}{3} L^3 = \tfrac{1}{3} ML^2$$

I about center, $L/2$:

$$I = \lambda \int_{-L/2}^{L/2} x^2 dx = \frac{2\lambda}{3}\left(\frac{L}{2}\right)^3 = \tfrac{1}{12} ML^2$$



Another way: the sum of two half-sticks pivoted at the end:

$$I = 2 \times \tfrac{1}{3}\left(\frac{M}{2}\right)\left(\frac{L}{2}\right)^2 = \tfrac{1}{12} ML^2$$