Physics 221

Department of Physics
The Citadel

S. Yost
October 2, 2009

Work and Energy – Part 2
Work in 2-3 Dimensions,
Changing Forces

Announcements

- Monday: problem set 7.
- Assigned Problems: Ch. 7:
  2, 7, 15, 16, 22, 31

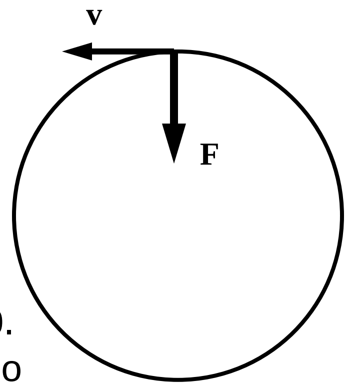
There is no "potential energy" in this set: we will cover that next week with Chapter 8.

- Today: Ch. 7: sec. 4 – 7

Topics: Work and kinetic energy in more than 1 dimension, the dot product, and work due to changing forces.

Work by Centripetal Force?

- What is the work done by the centripetal force in uniform circular motion?



The KE is constant: $K = \frac{1}{2} mv^2$.

Therefore, if the work-energy theorem still holds in two dimensions, the work is $W = 0$.

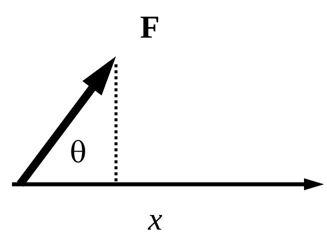
2-dimensional work is defined so that this is true: **a force perpendicular to the motion does no work.**

Work in More Dimensions

Consider a case where the displacement is along the x axis, but the force is not.

Only the component of the force parallel to the displacement changes the speed of an object, so only the parallel component of $\mathbf{F}$ contributes to work.

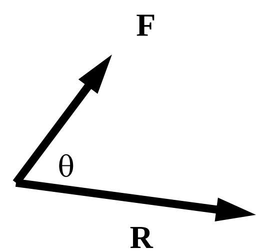
$$W = F_x x = Fx \cos \theta$$



Work in More Dimensions

In general, if the vector **R** has components x, y the work done by **F** when the particle moves along **R** is

$$W = FR\cos\theta$$
$$= F_x x + F_y y$$



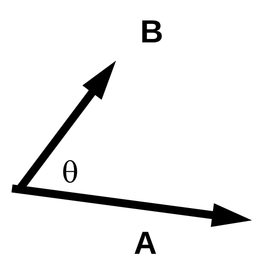
This is the sum of the 1d work done in each dimension.

Work in More Dimensions

This defines a type of product of two vectors which produces a number.

In general,

$$\mathbf{A} \cdot \mathbf{B} = AB\cos\theta$$
$$= A_x B_x + A_y B_y$$



is called the **dot product** or **scalar product** of **A** and **B**.

You can add z components the same way if needed.

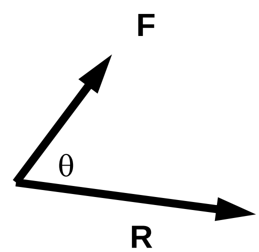
Work in More Dimensions

The work is the dot product of the force and displacement vectors:

$$W = F_x\, x + F_y\, y$$

$$= FR \cos\theta$$

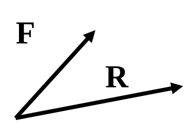
$$= \mathbf{F} \cdot \mathbf{R}$$



Work and Energy

General kinetic energy:

$$K = \tfrac{1}{2} mv^2 = \tfrac{1}{2} m(v_x^{\,2} + v_y^{\,2} + v_z^{\,2})$$



Work Energy Theorem:

$$\Delta K = W.$$

For a constant force:

$$W = F_x\, x + F_y\, y + F_z\, z = \mathbf{F} \bullet \mathbf{R}$$

The multi-dimensional Work-Energy theorem is obtained by adding up the work and kinetic energy in each of the dimensions separately.

Example

- A force **F** = 3N $\hat{\mathbf{i}}$ + 5N $\hat{\mathbf{j}}$ is applied while a particle is pushed with constant velocity **v** = 4 m/s $\hat{\mathbf{i}}$ **–** 3 m/s $\hat{\mathbf{j}}$**.** How much work is done by the force **F** while the particle moves 10 m?

$W = \mathbf{F}\cdot\mathbf{R}$, but what is **R**?

$\mathbf{R} = \mathbf{v}t$. $t = 2.0$ s.

$\mathbf{R} = 8\text{ m }\hat{\mathbf{i}} - 6\text{ m }\hat{\mathbf{j}}$.

$W = (3\text{N}\times 8\text{m}) - (5\text{N}\times 6\text{m}) = -6$ J.

Example

- What is the angle between the force and direction of motion?

$\mathbf{F}\cdot\mathbf{v} = Fv\cos\theta$.

$\mathbf{F}\cdot\mathbf{v}$ = 3x4 Nm/s – 5x3 Nm/s = – 3 Nm/s.

$F = (3^2 + 5^2)^{1/2}$ N = 5.83 N

$v = (3^2 + 4^2)^{1/2}$ m/s = 5 m/s.

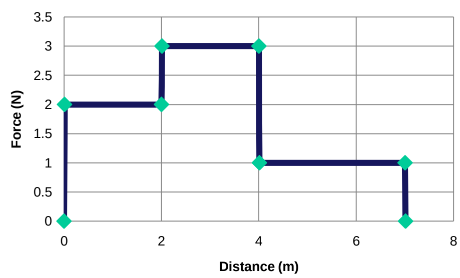
$\cos\theta = \mathbf{F}\cdot\mathbf{v} / Fv = -3/(5.83\times 5) = -0.1029$

$\theta = 96^o$.

Work is positive for acute angles, negative

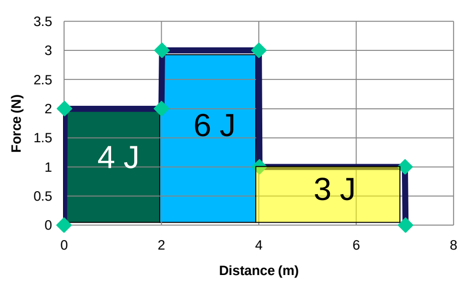
Work by a Changing Force

- If the force is not constant, but is constant in sections, then the total work is the sum of the work for each section. In this case,
- $W = (2\text{N})(2\text{m}) + (3\text{N})(2\text{m}) + (1\text{N})(3\text{m}) = 13$ J.



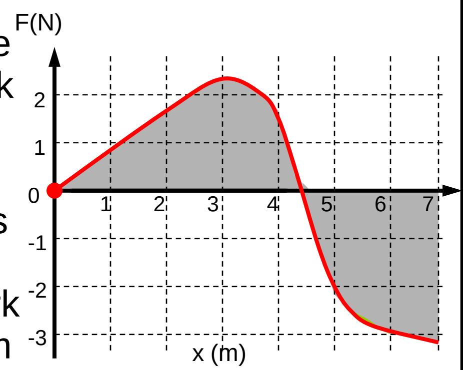
Work by a Changing Force

- Notice that the **work** is the **area** under the graph of the force as a function of distance: the sum of the three rectangles shown.
- This definition of the work can be generalized to a continuously changing force.



Work for a Changing Force

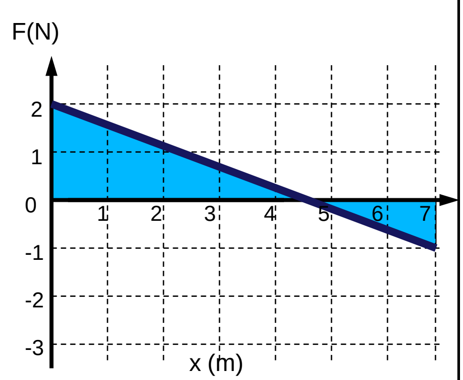
For any function $F(x)$, the area between the x axis and the curve $F(x)$ can be found, and gives the work done by the force over that distance.



Positive work of about 7 J is done between $x = 0$ and 4.5 m, while negative work of about -6 J is done from 4.5 to 7 m, for a net work of 1 J over all 7 m.

Work for a Changing Force

What is the work done by the force shown, which decreases from $+2$ N at $x = 0$ to -1 N at $x = 7$ m?



Find the area under the curve

$$F(x) = 2\text{N} - (3\text{N}/7\text{m})\ x.$$

The area above the axis is considered to be positive, and below the axis is negative.

Work for a Changing Force

Zero crossing at $x = 14/3$ m.

Positive work = area of triangle:

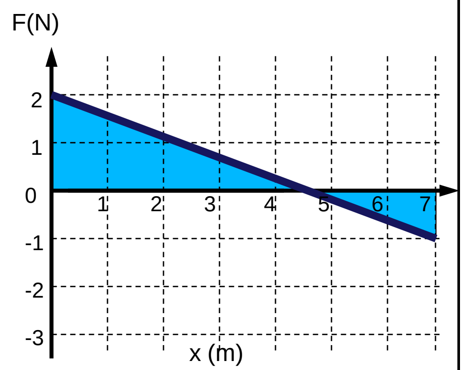
$W_1 = ½ (14/3)(2)J = 14/3$ J

Negative work:

$W_2 = - ½ (7/3)(1)J = -7/6$ J.

Total: $W = W_1 + W_2 = 7/2$ J

$= 3.5$ J.



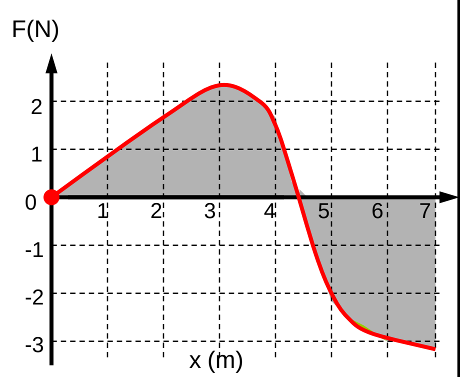
Work for a Changing Force

There is a name for this area: it is called the **integral**

$$W = \int F(x)\, dx$$

The upper and lower limits of x can be shown on the integral sign:

$$W = \int_0^{7m} F(x)\, dx$$



Fundamental Theorem of Calculus

- If $f(x)$ is the derivative of $g(x)$, so that $dg/dx = f(x)$, then the integral of $f(x)$ between $x = a$ and $x = b$ is

$$\int_a^b f(x)dx = g(b) - g(a).$$

- This means that finding the derivative and integral are inverse operations: $g(x)$ is called an antiderivative of $f(x)$.

Example

- If $f(x) = x^n$ find the integral of $f(x)$ from $x=a$ to $x=b$.

The derivative of x^{n+1} is $(n+1)x^n$, so $f(x)$ is the derivative of $g(x) = x^{n+1}/(n+1)$.

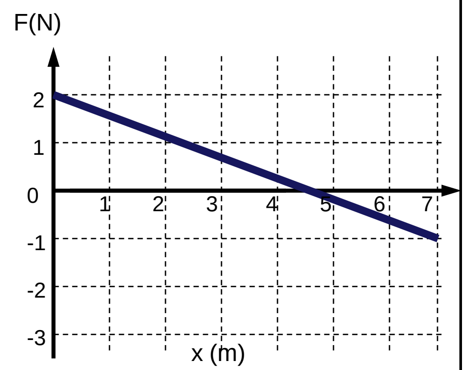
$$\int_a^b x^n dx = g(b) - g(0) = \frac{1}{n+1}\left(b^{n+1} - a^{n+1}\right)$$

This works for any n except $n = -1$.

Work for a Changing Force

Find the work done between $x = 0$ and 7 m for the force

$$F(x) = 2\text{N} - (3\text{N}/7\text{m})\, x.$$



$F(x)$ is the derivative of

$$G(x) = (2\text{N})x - (3\text{N}/14\text{m})x^2$$

$$W = \int_0^{7m} F(x)dx = G(7m) - G(0) = 14\text{J} - \frac{3\times 49}{14}\text{J} = 3.5\text{J}.$$

Work by a Spring

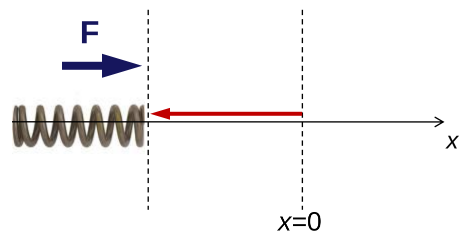
Springs push back with a force that increases with compression.

For an idealized spring,

$F = -kx$, if x is the compression from the equilibrium position.

Sign convention: compression is negative, extension is positive.

Equilibrium position $x=0$.

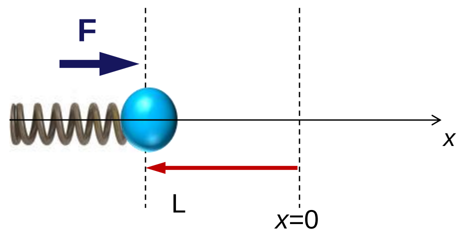


$F = -kx$ is called **Hooke's Law**. It is a descriptive law for an idealized spring.

k is called the **spring constant**.

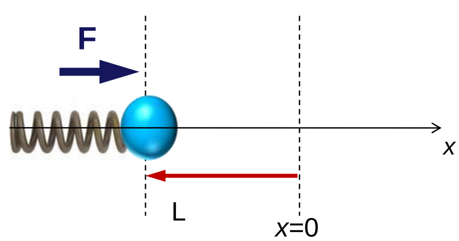
Work by a Spring

How much work does the spring do on an object if it is allowed to act on it while it returns to its equilibrium position after being compressed a total distance L?



Work by a Spring

Measure the position of the object in its direction of motion, with $x = 0$ at the uncompressed spring.



$$F(x) = -kx$$

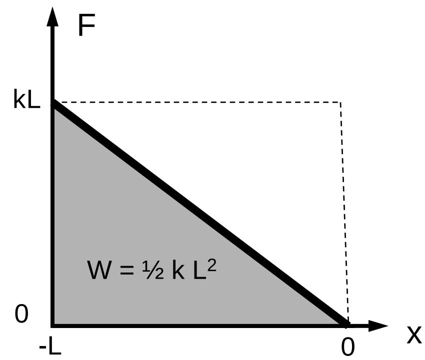
(since x<0)

$$W = \int_{-L}^{0} F(x)dx = -k\int_{-L}^{0} x dx = -\frac{k}{2}\left[(0)^2 - (-L)^2\right] = \frac{k}{2}L^2$$

Work by a Spring

Graphically, the work is the area under a plot of $F(x)$ on a graph of force as a function of position:

$$W = ½\, k\, L^2$$



General Work-Energy Theorem

We skipped this in class, but here is a proof that $W = \Delta K$ for a changing force.

For a general force in one dimension,

$$W = \int F(x)\, dx = m \int a(x)\, dx = m \int (dv/dt)\, dx$$

Use $dx = \frac{dx}{dv} dv$ and get

$$W = m\int \frac{dv}{dt}\frac{dx}{dv}\, dv = m\int \frac{dx}{dt} dv$$

Then $W = m \int (dx/dt)\, dv = m \int v dv$

$$= ½\, mv_f^2 - ½\, mv_i^2$$

The antiderivative of v Is ½ v^2.