PHYSICS 221 FINAL EXAM

Solutions - December 2009

Problem 1: [15pt] To start an avalanche on a mountain slope, an artillery shell is fired with an initial velocity of 330 m/s at 53.0° above the horizontal. It explodes on the mountainside 36.0 s after firing.

(a) [5pt] What are the x and y coordinates of the shell where it explodes, relative to its firing point?

$$v_{0x} = 330\frac{\text{m}}{\text{s}}\cos 53.0° = 198.6\frac{\text{m}}{\text{s}}, \quad v_{0y} = 330\frac{\text{m}}{\text{s}}\sin 53.0° = 263.5\frac{\text{m}}{\text{s}}.$$

$$x = v_{0x}t = 7150 \text{ m},$$
$$y = v_{0y}t - \frac{1}{2}gt^2 = 3136 \text{ m}.$$

(b) [5pt] What is the maximum height reached by the shell along its trajectory?

$$0 - v_{0y}^2 = -2gh,$$
$$h = \frac{v_{0y}^2}{2g} = 3542 \text{ m}.$$

(c) [5pt] What is the speed of the artillery shell when it strikes the mountainside?

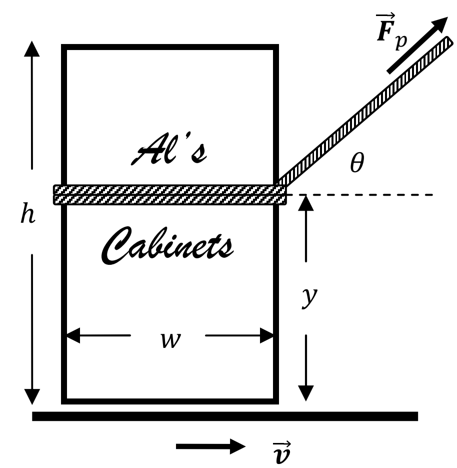
[If you don't have an answer to part (a), assume $y = 3000$ m.]

Using energy conservation,

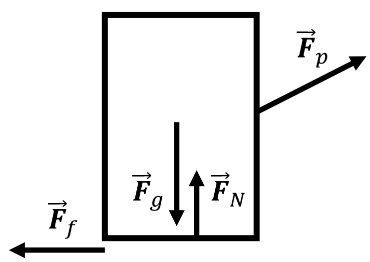
$$\frac{1}{2}mv_f^2 = \frac{1}{2}mv_0^2 - mgy,$$

$$v_f = \sqrt{v_0^2 - 2gy} = \sqrt{(330)^2 - 2(9.8)(3136)}\frac{\text{m}}{\text{s}} = 218\frac{\text{m}}{\text{s}}.$$

Problem 2: [15pt] A rectangular cabinet of weight $F_g = 400$ N is pulled across the floor at a constant speed $v = 1.5$ m/s by a person applying a force $F_p = 200$ N to a rope tied around the cabinet as shown, sloping upward at angle θ. There is a frictional force of $F_f = 160$ N between the cabinet and the floor.



(a) [5pt] Draw and label a free-body diagram for the cabinet, showing all forces acting on it.



(b) [5pt] What is the angle θ of the rope pulling the cabinet?

Balancing forces in the horizontal direction gives $F_p \cos\theta = F_f$. Then

$$\theta = \cos^{-1}\left(\frac{F_f}{F_p}\right) = \cos^{-1} 0.80 = 36.9°.$$

(c) [5pt] What is the coefficient of kinetic friction between the cabinet and the floor?

Balancing forces in the vertical direction shows that the normal force supporting the cabinet is

$$F_N = F_g - F_p \sin\theta = F_g - 0.60\,F_p = 280\text{ N}.$$

$$\mu_k = \frac{F_f}{F_N} = \frac{160}{280} = 0.571.$$

(d) [5pt] What is the power exerted by the person pulling the cabinet?

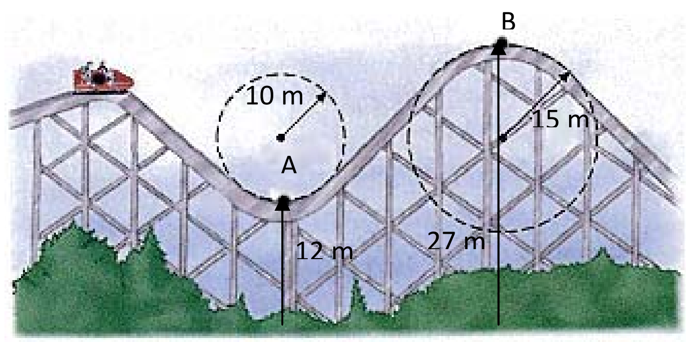
$$P = \vec{\boldsymbol{F}}_p \cdot \vec{\boldsymbol{v}} = F_p v \cos\theta = F_f v = (160\text{N})\left(1.5\,\frac{\text{m}}{\text{s}}\right) = 240\text{ W}.$$

(e) [5pt] If the cabinet's center of gravity is at its center, and its height and width are $h = 130$ cm, $w = 44$ cm, what is the maximum height y at which it can be pulled in this manner if it is not to tip over? [When the cabinet is about to tip, the contact forces with the floor act at its front edge.]

The torques about the bottom front edge are $\frac{1}{2}wF_g$ counterclockwise due to the weight acting at the center of mass, and $y\,F_p\cos\theta$ clockwise due to the force of the rope. (Only the horizontal component contributes because the lever arm is vertical.) Balancing these gives

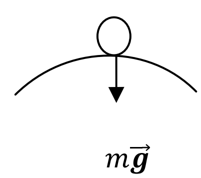
$$y = \frac{wF_g}{2F_p\cos\theta} = \frac{wF_g}{2F_f} = 1.25\,w = 55\text{ cm}.$$

Problem 3: [15pt] A roller coaster follows the track shown, where the height at point A is 12.0 m above the ground, and the height at point B is 27.0 m above the ground. The radius of curvature of the track is 10.0 m at point A and 15.0 m at point B.



(a) [5pt] What is the greatest speed the roller coaster car can have at point B for the passengers to stay in their seats without the help of a seat belt?

In the limiting case, the only force on the passengers is gravity. The acceleration is centripetal, with radius 15.0 m. Therefore,



$$g = \frac{v_B^2}{R_B}, \qquad v_B = \sqrt{gR_B} = \sqrt{147\,\frac{\text{m}^2}{\text{s}^2}} = 12.1\,\frac{\text{m}}{\text{s}}.$$

(b) [5pt] If the roller coaster has the speed at point B found in part (a), what is its speed at point A, neglecting friction and air resistance? [If you don't have an answer to part (a), assume it is 10 m/s.]

Energy conservation implies that $\frac{1}{2}mv_A^2 + mgh_A = \frac{1}{2}mv_B^2 + mgh_B$, so

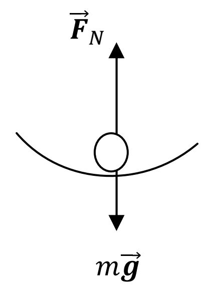
$$v_A = \sqrt{v_B^2 + 2g(h_B - h_A)} = \sqrt{147 + 294}\,\frac{\text{m}}{\text{s}} = \sqrt{441}\,\frac{\text{m}}{\text{s}} = 21.0\,\frac{\text{m}}{\text{s}}.$$

(c) [5pt] What is the normal force of the car's seat on a 75 kg passenger at point A in this case? [If you don't have an answer to part (b), you can assume the speed is 20 m/s.]

The net force is the mass times centripetal acceleration, so

$$F_N - mg = \frac{mv_A^2}{R_A}$$

$$F_N = m\left(g + \frac{v_A^2}{R_A}\right) = (75\text{ kg})\left(9.8 + \frac{441}{10}\right)\frac{\text{m}}{\text{s}^2} = 4043\text{ N}.$$



Problem 4: [15pt] A uniform cylindrical turntable of radius 2.00 m and mass 31.7 kg rotates counterclockwise in a horizontal plane with an initial angular speed of 4π rad/s. The fixed turntable bearing is frictionless. A lump of clay of mass 2.44 kg and negligible size is dropped onto the turntable from a small distance above it and immediately sticks to the turntable at a point 1.90 m to the east of the axis. [The moment of inertia of a uniform cylinder of radius R and mass M about an axis through the center is $\frac{1}{2}MR^2$.]

(a) [5pt] Find the final angular speed of the clay and turntable.

There is no external torque on the system of the clay lump and the turntable in this process, so angular momentum is conserved. The initial angular momentum about the axis of the turntable is $L_i = \frac{1}{2}MR^2\omega_i$ while the final angular momentum is $L_f = (\frac{1}{2}MR^2 + mr^2)\omega_f$ where M and m are the masses of the turntable and lump of clay, respectively, R is the radius of the turntable, and r is the radius at which the clay lands. Therefore, $L_i = L_f$ implies that

$$\frac{1}{2}M\omega_i = \left(\frac{1}{2}M + m\right)\omega_f, \quad \text{so that} \quad \omega_f = \frac{\omega_i}{1+2(mr^2/MR^2)} = \frac{4\pi}{1.139} = 11.0\,\frac{\text{rad}}{\text{s}}.$$

(b) [5pt] Is mechanical energy of the turntable-clay system conserved in this process? What, if any, is the change in energy?

No. This is not an elastic collision. Neglecting the height from which the lump of clay is dropped, since it is "small", the initial mechanical energy is the kinetic energy of the turntable, while the final mechanical energy is the kinetic energy of the turntable plus clay. The difference is the change in internal energy (vibrations or heat): $\Delta E = \frac{1}{4}MR^2\omega_i^2 - \frac{1}{2}\left(\frac{1}{2}MR^2 + mr^2\right)\omega_f^2 = \frac{1}{4}MR^2\omega_i^2 - \frac{1}{4}MR^2\omega_i\omega_f = \frac{1}{4}MR^2\omega_i(\omega_i - \omega_f) = \frac{1}{4}(31.7\text{ kg})(2.0\text{ m})^2(4\pi)(4\pi - 11.03)\text{J} = 611\text{ J}$. The second equality here uses conservation of angular momentum, from part (a).

(c) [5pt] Is momentum conserved in this process? What, if any, is the amount of impulse on the system imparted by the bearing when the clay lands?

No. The lump of clay is given a horizontal impulse to start it rotating, which must be imparted by the bearing supporting the turntable. The rotational speed of the lump of clay is $v = r\omega_f = (1.90\text{ m})\left(11.03\frac{\text{rad}}{\text{s}}\right) = 20.96\frac{\text{m}}{\text{s}}$. The impulse is $\Delta p = mv = (2.44\text{ kg})\left(20.96\frac{\text{m}}{\text{s}}\right) = 51.1\text{ Ns}$.

Problem 5. [15pt] A village maintains a large tank with an open top, containing water for emergencies. The water can drain from the tank through a hose of diameter 6.75 cm. The hose ends with a nozzle of diameter 2.50 cm. A rubber stopper is inserted into the nozzle. The water level in the tank is kept 7.50 m above the nozzle. [The density of water is 1 g/cm^3.]

(a) [5pt] Calculate the friction force exerted by the nozzle on the stopper.

The pressure behind the stopper is $P = \rho g h = \left(1000\frac{\text{kg}}{\text{m}^3}\right)\left(9.8\frac{\text{N}}{\text{kg}}\right)(7.50\text{ m}) = 7.35 \times 10^4\frac{\text{N}}{\text{m}^2}$. The area of the stopper is $A = \pi(1.25\text{ cm})^2 = 4.91\text{ cm}^2 = 4.91 \times 10^{-4}\text{m}^2$. Therefore, the force of friction holding the stopper in must be $F = PA = 36.1\text{ N}$.

(b) [5pt] How many kg/s of water flows from the pipe when the stopper is removed?

If the stopper is removed, Bernoulli's gives the flow velocity. At each point in the flow, $\frac{1}{2}\rho v^2 + \rho g h + P$ is constant. At the top of the tank, $P = 0, v = 0, h = 7.50\ m$. At the end of the nozzle, $P = 0$ and $h = 0$. Therefore, $\frac{1}{2}\rho v^2 = \rho g h$, giving $v = \sqrt{2gh} = \sqrt{147}\,\frac{\text{m}}{\text{s}} = 12.12\,\frac{\text{m}}{\text{s}}$. The volume rate of flow is $Av = 4.91\ \text{cm}^2 \left(1212\,\frac{\text{cm}}{\text{s}}\right) = 5951\,\frac{\text{cm}^3}{\text{s}}$ and the mass of 1000 cm^3 of water is 1 kg. Therefore, the mass rate of flow is 5.95 kg/s.

(c) [5pt] Calculate the gauge pressure of the flowing water in the hose just behind the nozzle.

Let point 1 be just inside the hose, and point 2 at the tip of the nozzle. The equation of continuity implies that $A_1 v_1 = A_2 v_2$, while Bernoulli's equation says that $P_1 + \frac{1}{2}\rho v_1^2 = P_2 + \frac{1}{2}\rho v_2^2$. The gauge pressure outside the hose is $P_2 = 0$, while v_2 was found in part (b). Using the equation of continuity to eliminate the unknown v_1 gives

$$P_1 = \frac{1}{2}\rho(v_2^2 - v_1^2) = \frac{1}{2}\rho v_2^2\left(1 - \left(\frac{A_2}{A_1}\right)^2\right) = \frac{1}{2}\left(1000\,\frac{\text{kg}}{\text{m}^3}\right)\left(147\,\frac{\text{m}^2}{\text{s}^2}\right)\left(1 - \left(\frac{2.50}{6.75}\right)^4\right)$$
$$= 72.1\ \text{kPa}.$$

Problem 6. [15pt] A 0.500 kg object attached to a spring with a force constant of 8.00 N/m vibrates in simple harmonic motion with an amplitude of 10.0 cm.

(c) [3pt] What is the period of the oscillation?

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{8.0}{0.5}}\,\frac{\text{rad}}{\text{s}} = 4.0\,\frac{\text{rad}}{\text{s}}.$$

$$T = \frac{2\pi}{\omega} = \frac{\pi}{2}\,\text{s} = 1.57\ \text{s}.$$

(b) [6pt] Calculate the maximum value (magnitude) of its speed and acceleration.

The projected circular motion model of simple harmonic motion gives (with the amplitude as the radius),

$$v_{\max} = A\omega = 40.0\,\frac{\text{cm}}{\text{s}}.$$

The maximum acceleration occurs when the spring produces the greatest force,

$$a_{\max} = \frac{F_{\max}}{m} = \frac{kA}{m} = 1.60\,\frac{\text{m}}{\text{s}^2}.$$

Projected circular motion gives the same result: $a_{\max} = A\omega^2 = 1.60\ \text{m/s}^2$.

(c) [6pt] Calculate the speed and acceleration when the object is 8.00 cm from the equilibrium position.

Energy conservation implies that $\frac{1}{2}kA^2 = \frac{1}{2}kx^2 + \frac{1}{2}mv^2$, so that

$$v = \sqrt{\frac{k}{m}(A^2 - x^2)} = \omega\sqrt{A^2 - x^2} = 4.0\ \text{s}^{-1}\sqrt{100 - 64}\ \text{cm} = 24.0\,\frac{\text{cm}}{\text{s}}.$$

Hooke's Law gives

$$a = -\frac{F}{m} = -\frac{kx}{m} = -1.28\,\frac{\text{m}}{\text{s}^2}.$$

Alternatively, you could use calculus. If $x = A\cos\omega t$, then the derivatives $\frac{d}{d\theta}\sin\theta = \cos\theta$, $\frac{d}{d\theta}\cos\theta = -\sin\theta$ with $\theta = \omega t$, together with the chain rule give

$$v = \frac{dx}{dt} = -A\sin\theta\,\frac{d\theta}{dt} = -A\omega\sin\theta = -v_{\max}\sin\theta,$$

$$a = \frac{dv}{dt} = -A\omega\cos\theta\,\frac{d\theta}{dt} = -x\omega^2.$$

At $x = 8.00\ \text{cm}$, $\sin\theta = \sin(\cos^{-1}0.800) = 0.600$, so the speed is 0.600 $v_{\max} = 24.0\,\frac{\text{m}}{\text{s}}$. The acceleration is $a = -(0.0800\ \text{m})(4.00\ \text{s}^{-1})^2 = -1.28\,\frac{\text{cm}}{\text{s}}$.