The Hypergeometric Representation of Feynman Diagrams and Construction of the Epsilon Expansion

S.A. Yost[⋆] and M. Kalmykov[†]

[⋆]Physics Department, The Citadel – The Military College of South Carolina

[†]Institut für Theoretische Physik, Universität Hamburg, Germany

with V.V. Bytev, B.A. Kniehl, and B.F.L. Ward

http://www.vic.com/syost/physics/papers/yost-ams.pdf

Overview

- Typically, each type of Feynman diagram (massless, single-scale, multileg, massive, *etc.*) has required a new technique.
- The Hypergeometric representation is a powerful tool applicable to many different kinds of diagrams. The universal properties stemming from the existence of this representation should be investigated.
- Dimensional regularization creates a need to characterize the coefficients of the expansion of hypergeometric functions about rational values of the parameters.
- Outline:
  - Representation of Feynman integrals
  - Integration-By-Parts (IBP) techniques
  - Hypergeometric representations
  - Differential reduction
  - Approaches to constructing the epsilon expansion

Feynman Diagrams: Basic Definitions

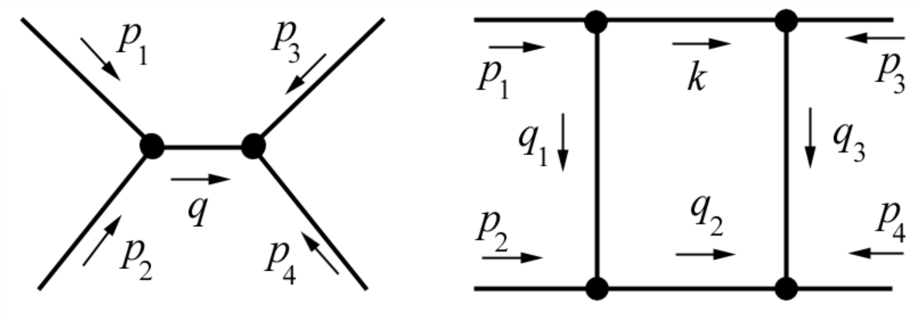
- Quantum field theory amplitudes are represented as a sum of **Feynman Diagrams**, graphs for which each edge (or **internal line**) and vertex is represented by a factor in a term of the quantum amplitude.
- The lines are interpreted as particles, parametrized by a mass m and a d-dimensional space-time momentum vector p.
- The vertices are 3- or 4-valent in renormalizable field theories.
- Each internal line is associated with a factor

$$\frac{1}{q_i^2 - m_i^2}$$

in the amplitude, called a **propagator**. $(q^2 = q_0^2 - q_1^2 - \cdots q_{d-1}^2.)$

- Feynman diagrams usually include half-edges, called **legs** or **external lines**, for which the momenta will be labeled p_i.
- There may also be a momentum-dependent numerator, depending on the particle type (scalar, spinor, vector).

Feynman Diagrams: Basic Definitions



- The first diagram is a tree. It would be associated with a propagator having momentum $q = p_1 + p_2 = -p_3 - p_4$.
- The second diagram contains a loop, in which one of the momenta of the internal lines (labeled k) is unconstrained and momentum conservation implies

$$q_1 = p_1 - k, \quad q_2 = p_1 + p_2 - k, \quad q_3 = p_3 + k, \quad p_1 + p_2 + p_3 + p_4 = 0.$$

- Integrating over all unconstrained momenta gives rise to a **Feynman Integral.** For L loops and n internal lines, and allowing the propagators to be raised to powers ν_j,

$$F_G = \int \prod_{r=1}^{L} d^d k_r \prod_{j=1}^{n} \frac{1}{(q_j^2 - m_j^2)^{\nu_j}}.$$

Feynman Integral: Schwinger representation

Using Schwinger's representation of the propagator,

$$\frac{1}{(q^2 - m^2)^j} = \frac{i^{-j}}{\Gamma(j)} \int_0^\infty d\alpha \alpha^{j-1} \exp\left[i\alpha(q^2 - m^2)\right]$$

and the relation

$$\int d^d k_1 \cdots d^d k_L \exp\left[i\left(\sum_{i,j} A_{i,j} k_i k_j + 2\sum_i r_i k_i\right)\right]$$

$$= e^{i\pi L(1-d/2)} \pi^{Ld/2} (\det A)^{-d/2} \exp\left[-i \sum_{i,j} A^{-1}_{i,j} r_i r_j\right],$$

the Feynman Integral can be written in the form

$$F_G \sim \prod_{k=1}^{n} \int_0^\infty \frac{d\alpha_k \alpha_k^{\nu_k - 1}}{\Gamma(\nu_k)} \frac{1}{U^{d/2}} \exp\left[i\left(\frac{Q}{U} - \sum_{l=1}^{n} \alpha_l m_l^2\right)\right].$$

Feynman Integral with Numerator

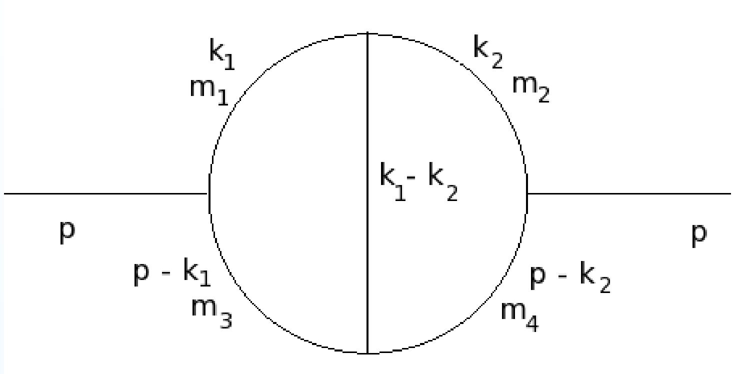
$$F_G \sim \prod_{k=1}^{n} \int_0^\infty \frac{d\alpha_k \alpha_k^{\nu_k - 1}}{\Gamma(\nu_k)} \frac{1}{U^{d/2}} \exp\left[i \left(\frac{Q}{U} - \sum_{l=1}^{n} m_l^2 \alpha_l \right) \right] .$$

- The functions U, Q are homogeneous functions of the parameters α of degree L and $L+1$, respectively. These functions can be derived from the topology of the corresponding Feynman graph G.
- A diagram with a numerator can be presented in a similar form by differentiating with respect to components of a vectors a:

$$\frac{q_{\mu_1} \cdots q_{\mu_n}}{(q^2 - m^2)^j} =$$

$$\frac{\partial}{\partial a_{\mu_1}} \cdots \frac{\partial}{\partial a_{\mu_n}} \frac{i^{-n-j}}{\Gamma(j)} \int_0^\infty d\alpha \alpha^{j-1} \exp\left[i\alpha(q^2 - m^2 + a \cdot q) \right] \Big|_{a=0} .$$

Example: 2 Loop Propagator Insertion



This diagram has 2 independent loop momenta k_1, k_2.

Construct α representation...

$$F_G = i^2 \left(\frac{\pi}{i}\right)^d \prod_{j=1}^{5} \frac{i^{-\nu_j}}{\Gamma(\nu_j)} \int_0^\infty \ldots \int_0^\infty \frac{d\alpha_j \alpha_j^{\nu_j - 1}}{[U(\alpha)]^{d/2}} \exp\left[i \left(\frac{Q(\alpha, a_1, a_2)}{U(\alpha)} - \sum_{l=1}^{5} \alpha_l m_l^2 \right) \right],$$

where

$$\begin{aligned} U(\alpha) &= \alpha_5(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) + (\alpha_1 + \alpha_3)(\alpha_2 + \alpha_4), \\ Q(\alpha, a_1, a_2) &= [(\alpha_1 + \alpha_2)(\alpha_3 + \alpha_4)\alpha_5 + \alpha_1\alpha_2(\alpha_3 + \alpha_4) + \alpha_3\alpha_4(\alpha_1 + \alpha_2)]p^2 \\ &\quad + (pa_1)Q_1 + (pa_2)Q_2 + a_1^2 Q_{11} + a_2^2 Q_{22} + (a_1 a_2)Q_{12}, \end{aligned}$$

with

$$\begin{aligned} Q_1 &= \alpha_3\alpha_5 + \alpha_4\alpha_5 + \alpha_2\alpha_3 + \alpha_3\alpha_4, \qquad Q_2 = \alpha_4\alpha_5 + \alpha_3\alpha_5 + \alpha_1\alpha_4 + \alpha_3\alpha_4, \\ Q_{11} &= -\frac{1}{4}(\alpha_2 + \alpha_4 + \alpha_5), \quad Q_{22} = -\frac{1}{4}(\alpha_1 + \alpha_3 + \alpha_5), \quad Q_{12} = -\frac{1}{2}\alpha_5. \end{aligned}$$

Dimensionally Regularized Feynman Integral

- The Feynman integral frequently diverges in the number of dimensions of interest.
- The **dimensionally degularized** Feynman integral is defined via this equation, treating the d as a complex number. Typically, $d = 4 - 2\varepsilon$.
- Dimensional regularization simultaneously renders UV (large momentum) and IR (small momentum, for massless particles) singularities finite.
- Integration by parts (IBP) can be applied to the regulated integrals:

$$\int d^d k \frac{\partial}{\partial k_\mu} G = 0 \ .$$

- This fact has been very useful in evaluating Feynman integrals.

F.V. Tkachov, Phys. Lett. **B100** (1981) 65;
K.G. Chetyrkin and F.V. Tkachov, Nucl. Phys. **B192** (1981) 159;
O.V. Tarasov, Phys. Rev. **D54** (1996) 6479

IBP Relations and Master Integrals

- **Integration by parts** leads to a set of recurrance relations among diagrams of a given topology but different powers of the propagators.
- The full set of recurrence relations should be solved by finding how the integral with powers of propagators $(j_1 + j_2 + \cdots + j_k)$ reduced to integrals with powers $(j_1 + j_2 + \cdots + j_k - 1)$
- The method involves taking derivatives of each integral with respect to momenta and reducing it to the original integral.
- The relations found permit a **reduction** to a basis set of **master integrals** in terms of which the diagrams of this class may be expressed.
- For new integrals which may appear within the reduction, the procedure is repeated.

Laporta, Int.J.Mod.Phys.**A15** (2000) 5087;
Anastasiou, Lazopoulos, JHEP **0407** (2004) 046
Smirnov, JHEP **0810** (2008) 107 [arXiv:0807.3243]
Studerus, Comp. Phys. Commun. **181** (2010) 1293 [arXiv:0912.2546]

IBP Example

One-loop Feynman integral with an arbitrary power of the propagator:

$$I_n = \int \frac{d^d k}{(k^2 - m^2)^n}.$$

The IBP identity

$$\int d^d k \frac{\partial}{\partial k_\mu} \left[\frac{k_\mu}{(k^2 - m^2)^n} \right] = 0$$

leads to a recurrance relation

$$(d - 2n) I_n - 2nm^2 I_{n+1} = 0$$

with solution

$$I_n = \frac{(-1)^n \Gamma\left(n + 1 - \frac{d}{2}\right)}{(n-1)! m^{2(n-1)} \Gamma\left(1 - \frac{d}{2}\right)} I_1$$

with a single master integral

$$I_1 = -i\pi^{d/2} m^{d-2} \Gamma\left(1 - \frac{d}{2}\right).$$

Feynman Integral: The Feynman Parametrization

- The **Feynman parametrization** is based on the identity

$$\prod_{i=1}^{n} \frac{1}{D_i^{\nu_i}} = \frac{\Gamma(\nu)}{\Gamma(\nu_1)\ldots\Gamma(\nu_n)} \prod_{i=1}^{n} \int_0^1 dx_i \, x_i^{\nu_i - 1} \frac{\delta(1 - x_1 \ldots - x_n)}{(x_1 D_1 + \ldots + x_n D_n)^{\nu}},$$

with $\nu = \sum_{i=1}^{n} \nu_i$.

- The Feynman Diagram can be reduced to the form

$$F_G \sim \prod_{k=1}^{n} \int_0^1 \frac{d\alpha_k \alpha_k^{\nu_k - 1}}{\Gamma(\nu_k)} \delta\left(\sum_l \alpha_l - 1\right) \frac{\Gamma\left(\nu - \frac{Ld}{2}\right) U^{\nu-(L+1)d/2}}{(U \sum_l m_l^2 \alpha_l - Q)^{\nu - Ld/2}} \, .$$

for some appropriate functions U and Q of the parameters, where $\nu = \sum_k \nu_k$.

The Mellin-Barnes Representation

The Mellin-Barnes representation relies on the identity

$$\frac{1}{(A+B)^{\lambda}} = \frac{1}{\Gamma(\lambda)} \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + \infty} dz \, \Gamma(-z) \, \Gamma(\lambda + z) \, \frac{B^z}{A^{\lambda+z}}.$$

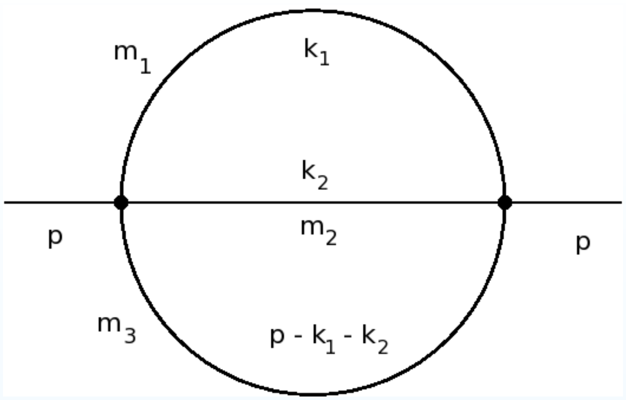
The contour is chosen to separate the poles in $\Gamma(-z)$ from the poles in $\Gamma(\lambda + z)$.

This relation is applied to the denominator in the Feynman Parametrization to break it up into monomials in the Feynman parameters x_i. The integration over the Feynman parameters can then be easily performed in terms of Γ functions,

$$\int_0^1 \prod_{i=1}^{n} dx_i \, x_i^{a_i - 1} \, \delta(1 - x_1 \ldots - x_n) = \frac{\Gamma(a_1) \ldots \Gamma(a_n)}{\Gamma(a_1 + \ldots + a_n)}.$$

The Mellin-Barnes representation is obtained by applying this procedure to all of the Feynman parameter integrals for a diagram.

Example: Sunset Diagram



$$F_G = \int \frac{d^d k_1 d^d k_2}{[(k_1 - p)^2 - m_1^2][k_2^2 - m_2^2][(k_1 - k_2)^2 - m_3^3]}$$

$$= \int_{-i\infty}^{i\infty} ds_1 ds_2 ds_3 \frac{m_1^{2s_1} m_2^{2s_2} m_3^{2s_3}}{(-p^2)^{s_1+s_2+s_3}} \Gamma(-s_1)\Gamma(-s_2)\Gamma(-s_3)$$

$$\Gamma(3-d+s_1+s_2+s_3) \frac{\Gamma(d/2-1-s_1)\Gamma(d/2-1-s_2)\Gamma(d/2-1-s_3)}{\Gamma(3d/2-3-s_1-s_2-s_3)}$$

$$\sim z_1^{d/2-1} z_2^{d/2-1} F_c^{(3)}(1, d/2, d/2, d/2, d/2; z_1, z_2, z_3)$$

$$-z_1^{d/2-1} \Gamma^2(1-d/2) F_c^{(3)}(1, 2-d/2, d/2, 2-d/2, d/2, z_1, z_2, z_3)$$

$$-z_2^{d/2-1} \Gamma^2(1-d/2) F_c^{(3)}(1, 2-d/2, d/2, 2-d/2, d/2, z_1, z_2, z_3)$$

$$-\Gamma(d/2-1)\Gamma(1-d/2)\Gamma(3-d) F_c^{(3)}(3-d, 2-d/2, 2-d/2, 2-d/2, d/2, z_1, z_2, z_3) \; ,$$

in terms of the hypergeometric function (in the case $n = 3$)

$$F_c^{(n)}(a, b; c_1, \cdots, c_n; z_1, \cdots z_n) = \sum_{k_1, \ldots k_n} \frac{(a)_{k_1+\cdots+k_n} (b)_{k_1+\cdots+k_n}}{(c_1)_{k_1} \cdots (c_n)_{k_n}} \frac{z_1^{k_1} \cdots z_n^{k_n}}{k_1! \cdots k_n!}$$

with arguments $z_1 = m_1^2/m_3^2, \quad z_2 = m_2^2/m_3^2, \quad z_3 = p^2/m_3^2$.

Epsilon Expansion

- Normally, the dimension is taken to be close to an integer, usually $d = 4 - 2\varepsilon$.
- The Feynman integral may be expressed as a Laurent series in ε, called the **epsilon expansion.**
- The final goal of the evaluation of a Feynman integral is to obtain an expression for all terms in the ε-expanded Feynman Integrals.

A good representation of the ε expansion should satisfy the following conditions:

- Stable numerical evaluation at the arbitrary values of agreements (the values of mass and external momenta).
- Analytical continuation in any region of values of physical parameters;
- Ability to construct the Laurent expansion at an arbitrary complex point;
- Ability to explicitly extract any logarithmic terms.

Hypergeometric Approach

Attempting to achieve these goals has motivated studying Feynman diagrams via hypergeometric functions. **Hypergeometric representation** of Feynman diagrams is motivated by a desire to achieve these goals.
The importance of hypergeometric functions to the study of Feynman diagrams is rooted in the following conjecture, which has yet to be proved rigorously.

Any Feynman diagram can be written as linear combination of Horn-type Hypergeometric Functions with rational parameters.

We are not aware of any exceptions to this statement.

Horn-Type Series Representation

Horn's definition: a Laurent series in r variables,

$$H(\vec{x}) = \sum C(\vec{m})\vec{x}^{m} \equiv \sum_{m_1, m_2, \cdots, m_r} C(m_1, m_2, \cdots, m_r) x_1^{m_1} \cdots x_r^{m_r},$$

is called **hypergeometric** if for each $i = 1, \cdots, r$ the ratio

$$\frac{C(\vec{m} + e_j)}{C(\vec{m})} = \frac{P_j(\vec{m})}{Q_j(\vec{m})}.$$

is a rational function in the index of summation: $\vec{m} = (m_1, \cdots, m_r)$, where $\vec{e}_j = (0, \cdots, 0, 1, 0, \cdots, 0)$, is unit vector with unity in the j^{th} place, and P_j, Q_j are polynomials.

Horn-type Hypergeometric Functions: Solution

Ore[1930] and Sato[1990] found the general form of the coefficients,

$$C(\vec{m}) = \prod_{i=1}^{r} \lambda_i^{m_i} R(\vec{m}) \left(\frac{\prod_{j=1}^{N} \Gamma(\mu_j(\vec{m}) + \gamma_j)}{\prod_{k=1}^{M} \Gamma(\nu_k(\vec{m}) + \delta_k)} \right) ,$$

where $N, M \geq 0$, $\lambda_j, \delta_j, \gamma_j \in \mathbb{C}$ are arbitrary complex numbers, $\mu_j, \nu_k : \mathbb{Z}^r \to \mathbb{Z}$ are arbitrary integer-valued linear maps, and R is an arbitrary rational function.
The Horn type hypergeometric function satisfies the following system of equations:

$$Q_j \left(\sum_{k=1}^{r} x_k \frac{\partial}{\partial x_k} \right) \frac{1}{x_j} H(\vec{x}) = P_j \left(\sum_{k=1}^{r} x_k \frac{\partial}{\partial x_k} \right) H(\vec{x}) .$$

Differential Reduction

Consider the hypergeometric series

$$H(\vec{\gamma};\vec{\sigma};\vec{x}) = \sum_{m_1,m_2,\cdots,m_r=0}^{\infty} \left(\frac{\prod_{j=1}^{K} \Gamma\left(\sum_{a=1}^{r} \mu_{ja} m_a + \gamma_j\right)}{\prod_{k=1}^{L} \Gamma\left(\sum_{b=1}^{r} \nu_{kb} m_b + \sigma_k\right)} \right) x_1^{m_1} \cdots x_r^{m_r} \; .$$

The lists $\vec{\gamma} = (\gamma_1, \cdots, \gamma_K)$ and $\vec{\sigma} = (\sigma_1, \cdots, \sigma_L)$ are called *upper* and *lower* parameters of the hypergeometric function, respectively.
Two functions with lists of parameters shifted by a unit, $\Phi(\vec{\gamma} + \vec{e_c}; \vec{\sigma}; \vec{x})$ and $\Phi(\vec{\gamma}; \vec{\sigma}; \vec{x})$, are related by a linear differential operator:

$$\begin{aligned} H(\vec{\gamma} + \vec{e_c}; \vec{\sigma}; \vec{x}) &= \left(\sum_{a=1}^{r} \mu_{ca} x_a \frac{\partial}{\partial x_a} + \gamma_c \right) H(\vec{\gamma}; \vec{\sigma}; \vec{x}) \\ H(\vec{\gamma}; \vec{\sigma} - \vec{e_c}; \vec{x}) &= \left(\sum_{b=1}^{r} \nu_{cb} x_b \frac{\partial}{\partial x_b} + \sigma_c - 1 \right) H(\vec{\gamma}; \vec{\sigma}; \vec{x}) \; . \end{aligned}$$

Horn-type Hypergeometric Functions: Takayama

The inverse differential operators can be constructed:

$$H(\vec{\gamma} - \vec{e_c}; \vec{\sigma}; \vec{x}) = \sum_a S_a(\vec{x}, \vec{\partial_x}) H(\vec{\gamma}; \vec{\sigma}; \vec{x})$$
$$H(\vec{\gamma}; \vec{\sigma} + \vec{e_c}; \vec{x}) = \sum_b L_b(\vec{x}, \vec{\partial_x}) H(\vec{\gamma}; \vec{\sigma}; \vec{x}) \; .$$

In this way, the Horn-type structure provides an opportunity to reduce hypergeometric functions to a set of basis functions with parameters differing from the original values by integer shifts:

$$P_0(\vec{x}) H(\vec{\gamma} + \vec{k}; \vec{\sigma} + \vec{l}; \vec{x}) = \sum_{m_1, \cdots, m_p = 0} P_{m_1, \cdots, m_r}(\vec{x}) \left(\frac{\partial}{\partial \vec{x}} \right)^{\vec{m}} H(\vec{\gamma}; \vec{\sigma}; \vec{x}) \; ,$$

where $P_0(\vec{x})$ and $P_{m_1, \cdots, m_p}(\vec{x})$ are polynomials with respect to $\vec{\gamma}, \vec{\sigma}$ and $\vec{x}$ and $\vec{k}, \vec{l}$ are lists of integers.

Example: Direct index-shifting operators

The generalized hypergeometric functions have the form

$$ {}_pF_q(\vec{a};\vec{b};z) \equiv {}_pF_q\left(\begin{array}{c|c} \vec{a} & \\ & z \\ \vec{b} & \end{array}\right) = \sum_{k=0}^{\infty} \frac{z^k}{k!} \frac{\prod_{i=1}^{p}(a_i)_k}{\prod_{j=1}^{q}(b_j)_k} \,, $$

where $(a)_k = \Gamma(a+k)/\Gamma(a)$ is called a *Pochhammer symbol*. The lists $\vec{a} = (a_1, \cdots, a_p)$ and $\vec{b} = (b_1, \cdots, b_q)$ are the upper and lower parameters of hypergeometric functions, respectively.

Direct index-shifting operators may be defined as follows:

$$ \begin{aligned} {}_pF_q(a_1+1,\vec{a};\vec{b};z) &= B^{+}_{a_1}\,{}_pF_q(a_1,\vec{a};\vec{b};z) \equiv \frac{1}{a_1}(\theta+a_1)\,{}_pF_q(a_1,\vec{a};\vec{b};z) \,, \\ {}_pF_q(\vec{a};b_1-1,\vec{b};z) &= H^{-}_{b_1}\,{}_pF_q(\vec{a};b_1,\vec{b};z) \equiv \frac{1}{b_1-1}(\theta+b_1-1)\,{}_pF_q(\vec{a};b_1,\vec{b};z) \,, \end{aligned} $$

where

$$ \theta = z\frac{d}{dz} \,. $$

Example: Inverse operators

For the special case ${}_{p+1}F_p$, inverse shifting operators satisfying

$$
\begin{aligned}
{}_{p+1}F_p(a_i - 1, \vec{a}; \vec{b}; z) &= B^-_{a_i}\ {}_{p+1}F_p(a_i, \vec{a}; \vec{b}; z) \ , \\
{}_{p+1}F_p(\vec{a}; b_i + 1, \vec{b}; z) &= H^+_{b_i}\ {}_{p+1}F_p(\vec{a}; b_1, \vec{b}; z) \ ,
\end{aligned}
$$

are found to be given by

$$
B^-_{a_i} = -\frac{a_i}{c_i}\left[t_i(\theta) - z \prod_{j \neq i} (\theta + a_j) \right]_- \ , \quad H^+_{a_i} = \frac{b_i - 1}{d_i}\left[\frac{d}{dz} \prod_{j \neq i} (\theta + b_j - 1) - s_i(\theta) \right]_+
$$

with

$$
\begin{aligned}
c_i &= -a_i \prod_{j=1}^{p} (b_j - 1 - a_i) \ , \quad t_i(x) = \frac{x \prod_{j=1}^{p} (x + b_j - 1) - c_i}{x + a_i} \\
d_i &= \prod_{j=1}^{p+1} (1 + a_j - b_i) \ , \quad s_i(x) = \frac{\prod_{j=1}^{p+1} (x + a_j) - d_i}{x + b_i - 1} ,
\end{aligned}
$$

and the $\pm$ subscripts on the brackets are shorthand indicating that $a_i \to a_i - 1$, $b_i \to b_i + 1$, inside the respective brackets.

Differential Reduction Example

In this way, any function ${}_{p+1}F_p(\vec{a}+\vec{m};\vec{b}+\vec{k};z)$ is expressible in terms of a single basic function and its first p derivatives:

$$S(a_i,b_j,z)\,{}_{p+1}F_p(\vec{a}+\vec{m};\vec{b}+\vec{k};z) = \left\{ R_1(a_i,b_j,z)\theta^p + R_2(a_i,b_j,z)\theta^{p-1} + \cdots \right.$$

$$\left. + R_p(a_i,b_j,z)\theta + R_{p+1}(a_i,b_j,z) \right\} {}_{p+1}F_p(\vec{a};\vec{b};z) \ ,$$

where m,k is the set of integer numbers, and S and R_i are polynomials in the parameters $\{a_i\},\{b_j\}$ and z.

Note: A Mathematica package HYPERDIRE has been implemented for the differential reduction of hypergeometric functions and is available at

https://sites.google.com/site/loopcalculations/home

Invariants of the Hypergeometric Representation

Is there a correlation between the number of master integrals for a given Feynman Diagram (via IBP) and number of basis elements in the differential reduction of Hypergeometric Functions?

Consider the standard hypergeometric representation of a Feynman diagram,

$$\Phi(d,\vec{j};\vec{z}) = \sum_{a=0}^{k} S_a(d,\vec{j},\vec{z})_{p+1+a}F_{p+a}(\vec{\beta}_a;\vec{\lambda}_a;\vec{\xi})$$

where $\vec{j}$ is a list of the powers of the propagators in the Feynman diagram, d is the space-time dimension, $\vec{\xi}$ are the arguments of the hypergeometric functions, which are related the kinematic invariants of the Feynman diagram, $\{\beta_a, \lambda_a\}$ are linear combinations of $\vec{j}$ and d with polynomial coefficients, and S_a are rational functions of the variables $\vec{z}$ with coefficients depending on d and $\vec{j}$.

Invariants of Hypergeometric Representation

Being a sum of holonomic functions, $\Phi(\vec{j};\vec{z})$ is also holonomic. Thus, the number of basis elements on the r.h.s. of the above equation is equal to the number of master-integrals $\Phi_k(\vec{z})$ that may be derived from the l.h.s. via IBP, giving, symbolically,

$$\Phi(d,\vec{j};\vec{z}) = \sum_{k=1}^{h} B_k(d,\vec{j};z)\Phi_k(d;z) \ .$$

The number h of nontrivial master integrals following from IBP which are not expressible in terms of gamma functions is then equal to the number of basis elements L for each term of r.h.s. of equation. Here, it is understood that diagrams that are expressible in terms of Gamma functions are not counted.

The number of basis elements in the framework of differential reduction is defined to be the highest power of the differential operator θ in

$${}_{p+1}F_p(\vec{A};\vec{B};z) = \sum_{l=0}^{v} P_l(z)\theta^l\, {}_{s+1}F_s(\vec{A}-\vec{I_1};\vec{B}-\vec{I_2};z) \ ,$$

where $\vec{I_1},\vec{I_2}$ are lists of integers and $P_l(z)$ are rational functions.

Invariants of Hypergeometric Representation

This analysis demonstrates that there is a very simple relation between the number h of nontrivial master integrals found from IBP (which are not expressible in terms of Gamma functions) and the maximal power v of θ generated by the differential reduction, namely

$$h = v + 1 \ .$$

This relation does not depend on the number k of hypergeometric functions entering original equation. This evidence leads to our **conjecture**:

Regardless of the type of functions in the r.h.s. of this equation,

the number of basis elements is the same

(up to a module of rational functions).

We are not aware of any exceptions to this statement, but it has not been proved in general.

Epsilon Expansion of a Hypergeometric Function

What does it mean to construct the ε expansion of a hypergeometric function?

A generalized hypergeometric function can be written as a series

$$ {}_PF_Q\left(\begin{array}{c} \{A_1 + a_1\varepsilon\}, \{A_2 + a_2\varepsilon\}, \cdots \{A_P + a_P\varepsilon\} \\ \{B_1 + b_1\varepsilon\}, \{B_2 + b_2\varepsilon\}, \cdots \{B_Q + b_q\varepsilon\} \end{array} \middle| z \right) = \sum_{j=0}^{\infty} \frac{z^j}{j!} \frac{\prod_{s=1}^{P} (A_s + a_s\varepsilon)_j}{\prod_{r=1}^{Q} (B_r + b_r\varepsilon)_j} \, , $$

where $(\alpha)_j \equiv \Gamma(\alpha + j)/\Gamma(\alpha)$ is the Pochhammer symbol.

Let us consider the Laurent expansion of a generalized hypergeometric function ${}_pF_{p-1}(\vec{A}; \vec{B}; z)$ with respect to its parameters.

Epsilon Expansion of a Hypergeometric Function

If each parameter list is shifted so that

$$\vec{A} \to \vec{A} + \varepsilon \vec{a}, \qquad \vec{B} \to \vec{B} + \varepsilon \vec{b},$$

then $\quad {}_pF_{p-1}(\vec{A} + \varepsilon\vec{a}; \vec{B} + \varepsilon\vec{b}; z) = {}_pF_{p-1}(\vec{A}; \vec{B}; z)$

$$+ \sum_{m_i, l_j = 1}^{\infty} \prod_{i=1}^{p} \prod_{j=1}^{p-1} \frac{(\varepsilon a_i)^{m_i}}{m_i!} \frac{(\varepsilon b_j)^{l_j}}{l_j!} \left(\frac{\partial}{\partial A_i} \right)^{m_i} \left(\frac{\partial}{\partial B_j} \right)^{l_j} {}_pF_{p-1}(\vec{A}; \vec{B}; z)$$

$$= {}_pF_{p-1}(\vec{A}; \vec{B}; z) + \sum_{m_i, l_j = 1} \prod_{i=1}^{p} \prod_{j=1}^{p-1} (\varepsilon a_i)^{m_i} (\varepsilon b_i)^{l_j} L_{\vec{A}, \vec{B}}(z) \ ,$$

$$= {}_pF_{p-1}(\vec{A}; \vec{B}; z) = {}_pF_{p-1}(\vec{A}; \vec{B}; z) + \sum_{k=1}^{\infty} \varepsilon^k L_{\vec{a}, \vec{b}, k}(z) \equiv \sum_{k=0}^{\infty} \varepsilon^k L_{\vec{a}, \vec{b}, k}(z) \ .$$

The goal of the ε expansion is to completely describe the coefficients $L_{\vec{a}, \vec{b}, k}(z)$.

Multiple Series Generated by the Epsilon Expansion

The ε expansion can be constructed in terms of **multiple harmonic sums.**

A useful starting point for expanding the gamma functions is

$$\ln \frac{\Gamma(k+1+\frac{p}{q}+j+z)}{\Gamma(k+1+\frac{p}{q}+j)} = \ln \frac{\Gamma(k+1+\frac{p}{q}+z)}{\Gamma(k+1+\frac{p}{q})} - \sum_{m=1}^{\infty} \frac{(-z)^m}{m} \sum_{r=1}^{j} \frac{1}{\left(r+k+\frac{p}{q}\right)^m}$$

$$= \ln \frac{\Gamma\left(1+\frac{p}{q}+z\right)}{\Gamma\left(1+\frac{p}{q}\right)} - \sum_{m=1}^{\infty} \frac{(-z)^m}{m} \sum_{r=1}^{j+k} \frac{1}{\left(r+\frac{p}{q}\right)^m} ,$$

In particular, for $p = 0$, we have

$$\ln \frac{\Gamma(1+j+z)}{\Gamma(1+z)} = \ln \Gamma(1+j) - \sum_{m=1}^{\infty} \frac{(-z)^m}{m} S_m(j) ,$$

where $S_a(j)$ is the harmonic sum defined as $S_a(j) = \sum_{k=1}^{j} \frac{1}{k^a}$.

Example: Gauss Hypergeometric Function

The ε expansion of the Gauss hypergeometric function gives

$$ {}_2F_1\left(\begin{array}{c} 1+a_1\varepsilon, 1+a_2\varepsilon \\ 2-\frac{p}{q}+c\varepsilon \end{array} \middle| z \right) = \frac{1}{z}\left(1-\frac{p}{q}+c\varepsilon\right) \sum_{j=1}^{\infty} z^j \frac{\Gamma(j)\Gamma\left(1-\frac{p}{q}\right)}{\Gamma\left(1-\frac{p}{q}+j\right)} \Delta \, , $$

where

$$ \Delta = \exp\left[\sum_{k=1}^{\infty} \frac{(-\varepsilon)^k}{k} \left(-A_k S_k(j-1) + c^k \mathrm{S}_k^{[q-p,q]}(j-1) \right) \right] \, , $$

with $A_k = a_1^k + a_2^k$. Here,

$$ \mathrm{S}_k^{[p,q]}(j) = \sum_{r=1}^{j} \frac{1}{\left(r+\frac{p}{q}\right)^k} $$

denotes the generalized multiple harmonic sum, which satisfies

$$ \mathrm{S}_k^{[p,q]}(j+1) = \mathrm{S}_k^{[p,q]}(j) + \frac{1}{\left(1+j+\frac{p}{q}\right)^k} \, . $$

Problem: How to evaluate these series?

The ε expansion of a hypergeometric function typically generates terms of the form

$$\sum_{j=1}^{\infty} z^j \frac{\Gamma\left(1-\frac{p}{q}\right)\Gamma(j)}{\Gamma\left(1-\frac{p}{q}+j\right)} \mathrm{S}_{a_1}^{[p_1,q_1]}(j-1)\cdots \mathrm{S}_{a_k}^{[p_k,q_k]}(j-1)$$

containing products of multiple harmonic sums

$$\mathrm{S}_k^{[p,q]}(j) = \sum_{r=1}^{j} \frac{1}{\left(r+\frac{p}{q}\right)^k} \,.$$

How can these be evaluated?

We will review some specific algorithms.

Expansion by Moch, Uwer, Weinzierl

Systematic algorithms for constructing the ε expansion of hypergeometric functions around integer values of the parameters has been developed by Moch, Uwer, and Weinzierl. These involve sums S_a and Z_a defined via

$$
\begin{aligned}
S_a(n;x) &= Z_a(n;x) = \sum_{k=1}^{n} \frac{x^k}{k^i} \\
S_{i,\vec{j}}(n;x_1,\cdots,x_l) &= \sum_{k=1}^{n} \frac{x_1^k}{k^i} S_{\vec{j}}(k;x_2,\cdots,x_l) \; , \\
Z_{i,\vec{j}}(n;x_1,\cdots,x_l) &= \sum_{k=1}^{n} \frac{x_1^k}{k^i} Z_{\vec{j}}(k-1;x_2,\cdots,x_l) \; ,
\end{aligned}
$$

These are useful for expanding the gamma function about integer values:

$$
\begin{aligned}
\Gamma(n+\varepsilon) &= \Gamma(1+\varepsilon)\Gamma(n) \left(1+\varepsilon Z_1(n-1)+\varepsilon^2 Z_{11}(n-1)+\varepsilon^3 Z_{111}(n-1)+\ldots\right) \; , \\
\Gamma(-n+1+\varepsilon) &= \frac{\Gamma(1+\varepsilon)}{\varepsilon} \frac{(-1)^{n-1}}{\Gamma(n)} \left(1+\varepsilon S_1(n-1)+\varepsilon^2 S_{11}(n-1)+\ldots\right) \; .
\end{aligned}
$$

When these algorithms fail...

There are some cases occuring in Feynman diagrams where the Moch, Uwer, Weinziel algorithm fails or does not apply...

- Integer values of parameters: Appell function F_4 and its generalizations;

$$F_4(a,b,c_1,c_2;x,y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{m+n}(b)_{m+n}}{(c_1)_m(c_2)_n}\frac{x^m}{m!}\frac{y^n}{n!}$$

- With one or more unbalanced rational parameter:

$$ {}_4F_3\left(\left. \begin{array}{c} 1, \frac{1}{2}+b_1\varepsilon, 1+a_1\varepsilon, 1+a_2\varepsilon \\ \frac{3}{2}+f_1\varepsilon, \frac{3}{2}+f_2\varepsilon \end{array} \right| z \right)$$

- With rational values of parameters: (2-loop sunset, equal masses)

$$ {}_2F_1\left(\left. \begin{array}{c} \frac{1}{3}+b_1\varepsilon, \frac{2}{3}+b_2\varepsilon \\ 2+c\varepsilon \end{array} \right| z \right)$$

Example: Rational Values of Parameters

An example of a Feynman diagram leading to rational values of parameters is

$$J_{122}^{(-)}(\alpha,\sigma_1,\sigma_2,m^2,M^2) = \int \frac{d^dk_1 d^dk_2}{[k_1^2-M^2]^{\sigma_1}[(k_1-k_2-q)^2-M^2]^{\sigma_2}[k_2^2-m^2]^{\alpha}}$$

(2-loop sunset, $m_1 = m_2 = M$, $m_3 = m$) when $p^2 = -m^2$, $d = 4 - 2\varepsilon$

$$J_{122}^{(-)}(1,1,1,m^2,M^2) = -(M^2)^{-\varepsilon}(m^2)^{1-\varepsilon}\frac{\Gamma^2(1+\varepsilon)}{\varepsilon^2(1-\varepsilon)}$$

$$\left[{}_4F_3\left(\begin{array}{c} 1, \frac{1}{2}, \frac{\varepsilon}{2}, \frac{1}{2}+\frac{\varepsilon}{2} \\ 2-\varepsilon, \frac{3}{4}, \frac{5}{4} \end{array} \middle| -\frac{m^4}{4M^4} \right) \right.$$

$$+ \left(\frac{M^2}{m^2}\right)^{1-\varepsilon} \frac{1}{(1-2\varepsilon)} {}_4F_3\left(\begin{array}{c} 1, -\frac{1}{2}+\frac{\varepsilon}{2}, \frac{\varepsilon}{2}, \varepsilon \\ \frac{3}{2}-\frac{\varepsilon}{2}, \frac{1}{4}+\frac{\varepsilon}{2}, \frac{3}{4}+\frac{\varepsilon}{2} \end{array} \middle| -\frac{m^4}{4M^4} \right)$$

$$\left. - \left(\frac{M^2}{m^2}\right)^{-\varepsilon} \frac{(1-\varepsilon)}{(2-\varepsilon)(1+2\varepsilon)} {}_4F_3\left(\begin{array}{c} 1, \frac{1}{2}+\frac{\varepsilon}{2}, \frac{1}{2}+\varepsilon, \varepsilon \\ 2-\frac{\varepsilon}{2}, \frac{3}{4}+\frac{\varepsilon}{2}, \frac{5}{4}+\frac{\varepsilon}{2} \end{array} \middle| -\frac{m^4}{4M^4} \right) \right] .$$

Generating Function Approach

This is an alternative to the MUW algorithms. Consider the multiple series:

$$\Sigma(z) = \sum_{j=1}^{\infty} z^j \eta(j) \ .$$

Let us find the recurrence relation for coefficients $\eta(j)$ with respect to the summation index j:

$$\sum_{m=0}^{k} P_{n_m}(m+j)\eta(m+j) = h(j)$$

where $P_{n_m}(x)$ is polynomial with respect to x of order n_m and $h(j)$ some inhomogeneous term. This difference equation can be converted into differential equation for generating function:

$$\sum_k Q_k(z) \frac{d}{dz} \Sigma(z) = H(z) \ ,$$

where

$$H(z) = \sum_{j=0}^{\infty} h(j) z^j \ .$$

Generating Function Approach: Example

$$\Sigma_{-;-;c}(z) = \sum_{j=1}^{\infty} \frac{z^j}{j^c}\eta(j) \quad \text{with} \quad \eta(j) = \frac{1}{\binom{2j}{j}}\frac{1}{j^c} \; .$$

Difference equation: $2(2j+1)(j+1)^{c-1}\eta(j+1) = j^c\eta(j)$.

Differential equation:

$$\left(\frac{4}{z}-1\right)\left(z\frac{d}{dz}\right)^c \Sigma_{-;-;c}(z) - \frac{2}{z}\left(z\frac{d}{dz}\right)^{c-1}\Sigma_{-;-;c}(z) = 1 \; , \qquad \Sigma_{-;-;c}(0) = 0.$$

New variable: $$y = \frac{\sqrt{z-4}-\sqrt{z}}{\sqrt{z-4}+\sqrt{z}} \; , \quad z = -\frac{(1-y)^2}{y} \; .$$

New System:

$$\left(-\frac{1-y}{1+y}y\frac{d}{dy}\right)^{c-1}\Sigma_{-;-;c}(z) = \frac{1-y}{1+y}\sigma_{-;-}(y) \; , \qquad y\frac{d}{dy}\sigma_{-;-}(y) = 1 \; .$$

Solution:

$$\Sigma_{-;-;1}(z) = \frac{1-y}{1+y}\ln y \; , \qquad \Sigma_{-;-;2}(z) = -\frac{1}{2}\ln^2 y \; , \ldots$$

$$\Sigma_{-;-;c}(z) = -\int_0^y dt \left[\frac{2}{1-t}+\frac{1}{t}\right]\Sigma_{-;-;c-1}(t) \; , \quad c > 2.$$

Multiple Polylogarithms (MPL)

By definition, the multiple polylogarithm is defined by power series

$$\mathrm{Li}_{k_1,k_2,\cdots,k_n}(x_1,x_2,\cdots,x_n) = \sum_{m_n>\cdots>m_1>0}^{\infty} \frac{x_1^{m_1}}{m_1^{k_1}} \frac{x_2^{m_2}}{m_2^{k_2}} \cdots \frac{x_n^{m_n}}{m_n^{k_n}} \, ,$$

where **weight** $k = k_1 + k_2 + \cdots + k_n$ and **depth** is equal to n.
It is defined for $|x_n| < 1$ and admit an analytical continuation.
The case $x_1 = \cdots = x_n = 1$ corresponds to multiple zeta values.
They satisfy **stuffle relations**:

$$\mathrm{Li}_m(x)\,\mathrm{Li}_n(y) = \mathrm{Li}_{m,n}(x,y) + \mathrm{Li}_{n,m}(y,x) + \mathrm{Li}_{m+n}(xy) \;\; ,$$

$$\begin{aligned} &\mathrm{Li}_{n_1,n_2}(x_1.x_2)\,\mathrm{Li}_{n_3}(x_3) = \\ &\quad \mathrm{Li}_{n_1,n_2,n_3}(x_1,x_2,x_3) + \mathrm{Li}_{n_1,n_3,n_2}(x_1,x_3,x_2) + \mathrm{Li}_{n_3,n_1,n_2}(x_3,x_1,x_2) \\ &\quad + \mathrm{Li}_{n_1+n_3,n_2}(x_1x_3,x_2) + \mathrm{Li}_{n_1,n_2+n_3}(x_1,x_2x_3) \;\; . \end{aligned}$$

Iterated Integrals

An iterated integral is defined as

$$\begin{aligned} I(z; a_k, a_{k-1} \cdots, a_1) &= \int_0^z \frac{dt}{t - a_k} I(t; a_{k-1}, \cdots, a_1) \\ &= \int_0^z \frac{dt_k}{t_k - a_k} \int_0^{t_k} \frac{dt_{k-1}}{t_{k-1} - a_{k-1}} \cdots \int_0^{t_2} \frac{dt_1}{t_1 - a_1} \end{aligned}$$

An important special case of this integral is

$$\begin{aligned} & G_{m_n, m_{n-1}, \cdots, m_1}(z; x_n, \cdots, x_1) \\ & \quad \equiv I(z; \underbrace{0, \cdots, 0}_{m_n - 1 \text{ times}}, x_n, \underbrace{0, \cdots, 0}_{m_{n-1} - 1 \text{ times}}, x_{n-1}, \cdots, \underbrace{0, \cdots, 0}_{m_1 - 1 \text{ times}}, x_1) \end{aligned}$$

The multiple polylogarithm is a special case of an iterated integral:

$$\begin{aligned} & \mathrm{Li}_{k_1, k_2, \cdots, k_n}(y_1, y_2, \cdots, y_n) \\ & \quad = (-1)^n G_{k_n, k_{n-1}, \cdots, k_2, k_1}\left(1; \frac{1}{y_n}, \frac{1}{y_n y_{n-1}}, \cdots, \frac{1}{y_1 \cdots y_n}\right) . \end{aligned}$$

Chen Iterated Integral

The integral $G_{z;m_k,m_{k-1}\ldots,m_1}(a_k,\ldots,a_1)$ is an iterated Chen integral w.r.t. the differential one-forms

$$\omega_0 = \frac{dy}{y}, \quad \omega_a = \frac{dy}{y-a},$$

where a is any number so that

$$G_{m_k,\ldots,m_1}(z;a_k,\ldots,a_1) = \int_0^z \omega_0^{m_k-1}\omega_{a_k} \times \cdots \times \omega_0^{m_1}\omega_{a_1} \ .$$

and

$$\mathrm{Li}_{k_1,\cdots,k_m}\left(\frac{1}{b_1},\ldots,\frac{1}{b_m}\right) = (-1)^m \int_0^1 \omega_0^{k_m-1}\omega_{b_m} \times \cdots \times \omega_0^{k_1-1}\omega_{b_m b_{m-1}\ldots b_1} \ .$$

The iterated integral representation gives rise to a “shuffle algebra” which is useful in the solution (as is the “stuffle algebra” noted earlier).

Special Cases of MPL

A special case of the multiple polylogarithm is the “generalized polylogarithm” defined by

$$\mathrm{Li}_{k_1,k_2,\ldots,k_n}(z) = \sum_{m_n > m_{n-1} > \cdots > m_1 > 0}^{\infty} \frac{z^{m_n}}{m_1^{k_1} m_2^{k_2} \cdots m_n^{k_n}}$$

where $|z| < 1$ when all $k_i \geq 1$, or $|z| \leq 1$ when $k_n \leq 2$.
Another special case is a “multiple polylogarithm of a square root of unity,” defined as

$$\mathrm{Li}_{\left(\substack{\sigma_1, \sigma_2, \cdots, \sigma_n \\ s_1, s_2, \cdots, s_n}\right)}(z) = \sum_{m_n > m_{n-1} > \cdots m_1 > 0} z^{m_n} \frac{\sigma_n^{m_n} \cdots \sigma_1^{m_1}}{m_n^{s_n} \cdots m_1^{s_1}} \; .$$

where $\vec{s} = (s_1, \cdots s_n)$ and $\vec{\sigma} = (\sigma_1, \cdots, \sigma_n)$ are multi-indices and σ_k belongs to the set of the square roots of unity, $\sigma_k = \pm 1$. This case has been analyzed in detail by Remiddi and Vermaseren, 2000.

Multiple Inverse Binomial Sums

$$\sum_{j=1}^{\infty} \frac{1}{\binom{2j}{j}} \frac{u^j}{j} S_{a_1}(j-1) \cdots S_{a_k}(j-1) \Bigg|_{u=-\frac{(1-y)^2}{y}} = \frac{1-y}{1+y} \sum_{p,\vec{s}} c_{p,\vec{s}} \ln^p y \mathrm{Li}_{\binom{\vec{\sigma}}{\vec{s}}}(y)$$

$$\sum_{j=1}^{\infty} \frac{1}{\binom{2j}{j}} \frac{u^j}{j^c} S_{a_1}(j-1) \cdots S_{a_k}(j-1) \Bigg|_{u=-\frac{(1-y)^2}{y}} = \sum_{p,\vec{s}} \tilde{c}_{p,\vec{s}} \ln^p y \mathrm{Li}_{\binom{\vec{\sigma}}{\vec{s}}}(y) \,, \quad c \geq 2 \,.$$

where c is a positive integer, $c_{p,\vec{s}}$ and $\tilde{c}_{p,\vec{s}}$ are rational coefficients, $S_a(j-1) = \sum_{i=1}^{j-1} \frac{1}{i^a}$, is harmonic sum and $\mathrm{Li}_{\binom{\vec{\sigma}}{\vec{s}}}(z)$ is the multiple polylogarithm of a square root of unity.

The result is the same for more general sums,

$$\sum_{j=1}^{\infty} \frac{1}{\binom{2j}{j}} \frac{u^j}{j^c} \sum_{k_1 > k_2 > \cdots k_m > 0}^{j-1} \frac{1}{m_1^{a_1} m_2^{a_2} \cdots m_k^{a_k}}$$

Multiple Binomial Sums

$$\sum_{j=1}^{\infty} \binom{2j}{j} u^j S_{a_1}(j-1)\cdots S_{a_k}(j-1)\Big|_{u=\frac{\chi}{(1+\chi)^2}} = \sum_{p,\vec{s}} \left[\frac{c_{p,\vec{s}}}{1-\chi} + d_{p,\vec{s}}\right] \ln^p \chi \, \mathrm{Li}_{\left(\vec{\sigma} \atop \vec{s}\right)}(\chi) \;,$$

$$\sum_{j=1}^{\infty} \binom{2j}{j} \frac{u^j}{j^c} S_{a_1}(j-1)\cdots S_{a_k}(j-1)\Big|_{u=\frac{\chi}{(1+\chi)^2}} = \sum_{p,\vec{s}} \tilde{c}_{p,\vec{s}} \ln^p \chi \mathrm{Li}_{\left(\vec{\sigma} \atop \vec{s}\right)}(\chi) \;, \quad c \geq 1$$

where c is a positive integer, $c_{p,\vec{s}}$, $\tilde{c}_{p,\vec{s}}$ and $d_{p,\vec{s}}$ are rational coefficients, $\mathrm{Li}_{\left(\vec{\sigma} \atop \vec{s}\right)}(z)$ is the multiple polylogarithm of a square root of unity and $S_a(j-1) = \sum_{i=1}^{j-1} \frac{1}{i^a}$.

The result is the same for more general sums,

$$\sum_{j=1}^{\infty} \binom{2j}{j} \frac{u^j}{j^c} \sum_{k_1 > k_2 > \cdots k_m > 0}^{j-1} \frac{1}{m_1^{a_1} m_2^{a_2} \cdots m_k^{a_k}}$$

General Case I

$$\sum_{j=1}^{\infty} z^j \frac{\Gamma(j)\Gamma\left(1-\frac{p}{q}\right)}{\Gamma\left(1+j-\frac{p}{q}\right)} S_{a_1}(j-1)S_{a_2}(j-1)\cdots S_{a_k}(j-1)\Bigg|_{z=z(\xi)}$$

$$= \xi^p \sum_{\vec{J},\vec{s}} c_{\vec{J},\vec{s}} \mathrm{Li}_{\vec{s}}\left(\lambda_q^{j_1-j_2}, \lambda_q^{j_2-j_3}, \cdots, \lambda_q^{j_{r-1}-j_r}, \lambda_q^{j_r}\xi\right) ,$$

where p, q are arbitrary integers, $\xi = \left(\frac{z}{z-1}\right)^{1/q}$, $\lambda_q = \exp\left(i\frac{2\pi}{q}\right)$, and

$$1 \le \{j_m\} \le q \ , \quad \sum_{k=1}^{r} s_k = 1+a_1+\cdots+a_p \ ,$$

p, q are arbitrary integers.

The result is the same for more general sums:

$$S_{a_1}(j-1)\cdots S_{a_k}(j-1) \to \sum_{k_1>k_2>\cdots k_m>0}^{j-1} \frac{1}{m_1^{a_1} m_2^{a_2}\cdots m_k^{a_k}}$$

General Case II

$$\sum_{j=1}^{\infty} \frac{z^j}{j^c} \frac{\Gamma(j)\Gamma\left(1-\frac{p}{q}\right)}{\Gamma\left(1+j-\frac{p}{q}\right)} S_{a_1}(j-1) S_{a_2}(j-1) \cdots S_{a_k}(j-1) \Big|_{z=z(\xi)}$$

$$= \sum_{\vec{J},\vec{s}} \tilde{c}_{\vec{J},\vec{s}} \mathrm{Li}_{\vec{s}} \left(\lambda_q^{j_1-j_2}, \lambda_q^{j_2-j_3}, \cdots, \lambda_q^{j_{r-1}-j_r}, \lambda_q^{j_r}\xi \right) \qquad (c \geq 1) \, ,$$

where

$$1 \leq \{j_m\} \leq q \, , \quad \sum_{k=1}^{r} s_k = 1+c+a_1+\cdots+a_p \, .$$

The result is again the same for more general sums:

$$S_{a_1}(j-1) \cdots S_{a_k}(j-1) \to \sum_{k_1>k_2>\cdots k_m>0}^{j-1} \frac{1}{m_1^{a_1} m_2^{a_2} \cdots m_k^{a_k}}$$

Examples

$$\sum_{j=1}^{\infty} z^j \frac{\Gamma\left(1-\frac{p}{q}\right)\Gamma(j)}{\Gamma\left(1-\frac{p}{q}+j\right)} = -\xi^p \sum_{j_1=1}^{q} \lambda_q^{j_1 p} \mathrm{Li}_1\left(\lambda_q^{j_1}\xi\right) ,$$

$$\sum_{j=1}^{\infty} z^j \frac{\Gamma\left(1-\frac{p}{q}\right)\Gamma(j)}{\Gamma\left(1+j-\frac{p}{q}\right)} S_1(j-1) = \xi^p \sum_{j_1,j_2=1}^{q} \lambda_q^{j_1 p} \mathrm{Li}_{1,1}\left(\lambda_q^{j_1-j_2}, \lambda_q^{j_2}\xi\right) ,$$

$$\sum_{j=1}^{\infty} z^j \frac{\Gamma\left(1-\frac{p}{q}\right)\Gamma(j)}{\Gamma\left(1+j-\frac{p}{q}\right)} \mathrm{S}_1^{[q-p,q]}(j-1) = -\xi^p q \sum_{j_1=1}^{q} \lambda_q^{j_1 p} \mathrm{Li}_2\left(\lambda_q^{j_1}\xi\right) ,$$

$$\sum_{j=1}^{\infty} z^j \frac{\Gamma\left(1-\frac{p}{q}\right)\Gamma(j)}{\Gamma\left(1+j-\frac{p}{q}\right)} \left(\left[\mathrm{S}_1^{[q-p,q]}(j-1)\right]^2 + \mathrm{S}_2^{[q-p,q]}(j-1)\right) = -2q^2\xi^p \sum_{j_1=1}^{q} \lambda_q^{j_1 p} \mathrm{Li}_3\left(\lambda_q^{j_1}\xi\right) ,$$

$$\sum_{j=1}^{\infty} z^j \frac{\Gamma\left(1-\frac{p}{q}\right)\Gamma(j)}{\Gamma\left(1+j-\frac{p}{q}\right)} S_1(j-1)\mathrm{S}_1^{[q-p,q]}(j-1) =$$

$$q\xi^p \sum_{j_1,j_2=1}^{q} \lambda_q^{j_1 p} \left\{ \mathrm{Li}_{1,2}\left(\lambda_q^{j_1-j_2}, \xi\lambda_q^{j_2}\right) + \mathrm{Li}_{2,1}\left(\lambda_q^{j_1-j_2}, \xi\lambda_q^{j_2}\right) \right\} ,$$

$\ldots$

Direct Solution of the System of Differential Equations

Recall that any multiple series generated by the ε expansion is related to derivatives of Horn-type hypergeometric functions with respect to parameters. Horn hypergeometric functions satisfy a system of differential equations. Instead of starting with the difference equation for multiple sums, it is thus possible to construct the ε expansion for Horn-type functions directly.

$$\Sigma^{(k)}_{\vec{a};\vec{b};c}(z) = \sum_{s,\vec{\alpha},\vec{\beta}} c_s \left(\partial/\partial\vec{A}\right)^{\vec{\alpha}_s} \left(\partial/\partial\vec{B}\right)^{\vec{\beta}_s} {}_{p+s}F_{p-1+s}\left(\vec{A}_s;\vec{B}_s;z\right)\Big|_{\vec{A}_s=\vec{m}_s;\vec{B}_s=\vec{n}_s} ,$$

where, for example:

$$\Sigma^{(k)}_{a1,\cdots,a_p;b_1,\cdots,b_q;c}(z) = \sum_{j=1}^{\infty} \frac{1}{\binom{2j}{j}^k} \frac{z^j}{j^c} S_{a_1}(j-1)\cdots S_{a_p}(j-1) S_{b_1}(2j-1)\cdots S_{b_q}(2j-1) ,$$

where $S_a(n) = \sum_{j=1}^{n} 1/j^a$ is the harmonic sum and $k = \pm 1$.

Direct Solution of the System of Differential Equations

Consider the ε expansion of a hypergeometric function $\omega(z) = {}_pF_{p-1}\left(\vec{a}\varepsilon, A+c\varepsilon; \vec{1}+\vec{b}\varepsilon, B+f\varepsilon; z\right)$. Starting from the differential equation

$$\left[z\,(\theta+A+c\varepsilon)\prod_{j=1}^{p-1}(\theta+a_j\varepsilon)-\theta\,(\theta+B-1+f\varepsilon)\prod_{k=1}^{p-2}(\theta+b_k\varepsilon)\right]\omega(z) = 0\;,$$

we obtain equations for the coefficients of its ε expansion $\omega(z) = 1 + \sum_{j=1}^{\infty} w_k(z)\varepsilon^k$:

$$\left[(1-z)\frac{d}{dz}+\frac{B-1}{z}-A\right]\theta^{p-1}w_m(z) = \left[P_1^{(p)}(\vec{a},c)-\frac{1}{z}P_1^{(p-1)}(\vec{b},f)\right]\theta^{p-1}w_{m-1}(z)$$

$$+\sum_{j=2}^{p-1}\left[P_j^{(p)}(\vec{a},c)-\frac{1}{z}P_j^{(p-1)}(\vec{b},f)\right]\theta^{p-j}w_{m-j}(z)+AP_{p-1}^{(p-1)}(\vec{a})w_{m-p+1}(z)$$

$$+\sum_{k=1}^{p-2}\left[AP_k^{(p-1)}(\vec{a})-\frac{(B-1)}{z}P_k^{(p-2)}(\vec{b})\right]\theta^{p-1-k}w_{m-k}(z)+P_p^{(p)}(\vec{a},c)w_{m-p}(z)\;,$$

where $\theta = zd/dz$ and the polynomials $P_j^{(p)}(r_1,\cdots,r_p)$ are

$$\prod_{k=1}^{p}(z+r_k) = \sum_{j=0}^{p}P_{p-j}^{(p)}(r_1,\cdots,r_p)z^j \equiv \sum_{j=0}^{p}P_{p-j}^{(p)}(\vec{r})z^j = \sum_{j=0}^{p}P_j^{(p)}(\vec{r})z^{p-j}\;.$$

Solution: Finite Part

The first non-vanishing term corresponds to $m = p$ if $A = 0$, and to $m = p - 1$ otherwise. In both cases, the main equation reduces to

$$\left[(1-z)\frac{d}{dz} + \frac{B-1}{z} - A\right]\theta^{p-1} w_{p-1+\delta_{A,0}}(z) = (A + c\delta_{A,0}) P_{p-1}^{(p-1)}(\vec{a}) \ .$$

Let us redefine the higher derivatives of $\omega(z)$ using new functions:

$$\theta^{p-1} w_k(z) \to h(z)\theta^{p-1}\phi_k(z).$$

where $\phi_k(z)$ is a new function and

$$h(z) = (-1)^A z^{1-B}(z-1)^{B-A-1} \ ,$$

with A and B being arbitrary rational numbers. Then, we have

$$(-1)^{A-1} z^{-B}(z-1)^{B-A}\theta^p \phi_{p-1+\delta_{A,0}}(z) = (A + c\delta_{A,0}) P_{p-1}^{(p-1)}(\vec{a}) \ .$$

Solution: Finite Part

The solution of this equation can be written as a multiply iterated integral,

$$\phi^{(p-1)}_{p-1+\delta_{A,0}(z)} \sim \int_0^z \frac{dt_1}{t_1} \int_0^{t_1} \frac{dt_2}{t_2} \cdots \int_0^{t_{p-1}} \frac{dt_p}{t_p} \frac{t_p^B}{(t_p-1)^{B-A}} ,$$

where the constant part is omitted for simplicity.
This solution can be written in terms of multiple polylogarithms (hyperlogarithms) defined as iterative integrals over rational one-forms,

$$I_k(z; a_k, a_{k-1}, \ldots, a_1) = \int_0^z \frac{dt}{t-a_k} I_{k-1}(t; a_{k-1}, \ldots, a_1) ,$$

where z is the argument, $\{a_i\}$ is the set of parameters, and k is the weight of the hyperlogarithm. In this way, the solution may be expressed in terms of hyperlogarithms if a parametrization $z \to \xi(z)$ exists such that following two conditions are fulfilled for some rational functions $Q(\xi)$, $R(\xi)$:

$$\frac{dz}{(1-z)h(z)} = Q(\xi) d\xi , \qquad \frac{dz}{z} = R(\xi) d\xi .$$

Solution: Epsilon Parts

To analyze the structure of the highest coefficients of the ε expansions, let us consider the original function $\omega(z)$ and its first $p-1$ derivatives as independent functions, $f^{(k)} = (\omega, \theta\omega, \cdots, \theta^{p-1}\omega)$, $k = 0, \cdots, p-1$.

Each $f^{(k)}$ has a ε expansion $f^{(k)}(z) = \sum_{j=0}^{\infty} f_j^{(k)}(z)\varepsilon^j$ with boundary conditions $f_0^{(0)}(z) = 1$ and $f_j^{(k)}(0) = 0$, $j \geq 1$, $k = 1, \cdots, p-1$. Defining $\theta^{p-1}\omega_k(z) = h(z)\phi_j^{(p-1)}(z)$, we can convert the original equation into a system of first-order differential equations,

$$h(z)(1-z)\frac{d}{dz}\phi_m^{(p-1)}(z) = h(z)\left[P_1^{(p)}(\vec{a},c) - \frac{1}{z}P_1^{(p-1)}(\vec{b},f)\right]\phi_{m-1}^{(p-1)}(z)$$

$$+ \sum_{j=2}^{p-1}\left[P_j^{(p)}(\vec{a},c) - \frac{1}{z}P_j^{(p-1)}(\vec{b},f)\right] f_{m-j}^{(p-j)}(z) + AP_{p-1}^{(p-1)}(\vec{a})w_{m-p+1}(z)$$

$$+ \sum_{k=1}^{p-2}\left[AP_k^{(p-1)}(\vec{a}) - \frac{(B-1)}{z}P_k^{(p-2)}(\vec{b})\right] f_{m-k}^{(p-1-k)}(z) + P_p^{(p)}(\vec{a},c)w_{m-p}(z) \,,$$

$$\theta f_m^{(p-2)}(z) = h\phi_m^{(p-1)}(z) \,,$$

$$\theta f_m^{(j-1)}(z) = f_m^{(j)}(z) \quad \text{for} \qquad j = 1, \cdots, p-2 \,.$$

Solution: Epsilon Parts

- The solution of this system can again be presented as an iterated integral over a rational one-form, if two additional conditions are satisfied:

$$\frac{dz}{z}\frac{1}{h(z)} = P_1(\xi)d\xi \; , \qquad \frac{dz}{z}h(z) = P_2(\xi)d\xi \; .$$

  where P_1 and P_2 are rational functions.

- As a consequence of the universality of hyperlogarithms, any iterated integral over a rational function may be expressed again in terms of hyperlogarithms. It is easy to show that the two equations are not functionally independent. In fact, we obtain

$$R^2(\xi) = P_1(\xi)P_2(\xi) \; , \quad h(z) = \frac{R(\xi)}{P_1(\xi)} = \frac{P_2(\xi)}{R(\xi)} \; .$$

Example: Gauss Hypergeometric Function

Let us consider as the basis the Gauss hypergeometric function with the following set of parameters:

$$\omega(z) = {}_2F_1\left(\frac{p_1}{q_1} + a_1\varepsilon, \frac{p_2}{q_2} + a_2\varepsilon; 1 - \frac{p_3}{q_3} + c\varepsilon; z\right) .$$

It is the solution of the differential equation

$$\left(z\frac{d}{dz} + \frac{p_1}{q_1} + a_1\varepsilon\right)\left(z\frac{d}{dz} + \frac{p_2}{q_2} + a_2\varepsilon\right)\omega(z) = \frac{d}{dz}\left(z\frac{d}{dz} - \frac{p_3}{q_3} + c\varepsilon\right)\omega(z) ,$$

with boundary conditions $\omega(0) = 1$ and $z\frac{d}{dz}\omega(z)\Big|_{z=0} = 0$.
Due to the analyticity of the Gauss hypergeometric function with respect to its parameters, this equation is valid in each order of ε, i.e. it holds for every coefficient function $\omega_k(z)$ in the expansion

$$\omega(z) = \sum_{k=0}^{\infty} \omega_k(z)\varepsilon^k .$$

Example: Gauss Hypergeometric Function

The boundary conditions for the coefficient functions are

$$\begin{aligned} \omega_k(z) &= 0 && (k < 0) \, , \\ \omega_k(0) &= 0 && (k \geq 1) \, , \\ z \tfrac{d}{dz} \omega_k(z) \Big|_{z=0} &= 0 && (k \geq 0) \, . \end{aligned}$$

the original equation can be rewritten in terms of the coefficients functions ω_k as

$$\left[(1-z) \frac{d}{dz} - \left(\frac{p_1}{q_1} + \frac{p_2}{q_2} \right) - \frac{1}{z} \frac{p_3}{q_3} \right] z \frac{d}{dz} \omega_k - \frac{p_1 p_2}{q_1 q_2} \omega_k$$
$$= \left(a_1 + a_2 - \frac{c}{z} \right) z \frac{d}{dz} \omega_{k-1} + \left(a_1 \frac{p_2}{q_2} + a_2 \frac{p_1}{q_1} \right) \omega_{k-1} + a_1 a_2 \omega_{k-2} \, .$$

Example: Gauss Hypergeometric Function

The **main idea** of our approach is to find a new parametrization, through a change of variables $z \to \xi(z)$, and defining new functions $\rho_k(\xi)$, related to the first derivative of the original functions $\omega_k(\xi)$ as

$$\rho_k(\xi) = \sum_j \Gamma_{kj}(\xi) \frac{d}{d\xi} \omega_j(\xi) \ ,$$

so that original equation can be rewritten as a system of first-order linear differential equations with rational coefficients:

$$\begin{aligned} \frac{d}{d\xi}\omega_k(\xi) &= \rho_k(\xi) \sum_j \frac{A_j}{\xi - \alpha_j} \ , \\ \frac{d}{d\xi}\rho_k(\xi) &= \rho_{k-1}(\xi) \sum_j \frac{B_j}{\xi - \beta_j} + \omega_{k-1}(\xi) \sum_j \frac{C_j}{\xi - \gamma_j} + \omega_{k-2}(\xi) \sum_j \frac{D_j}{\xi - \sigma_j} \ , \end{aligned}$$

where $A_j, B_j, C_j, D_j, \alpha_j, \beta_j, \gamma_j, \sigma_j \in \mathbb{C}$ ($j = 1, 2, \cdots$). Then, the iterative solution of this system can be constructed.

When $\omega_0(z) = 1$ ($\rho_0(z) = 0$), this solution can be expressed in terms of hyperlogarithms depending on the parameters $\alpha_j, \beta_j, \gamma_j, \sigma_j$, possibly times powers of logarithms.

Example: Gauss Hypergeometric Functions

These methods have been extended to rational values of the parameters as well, leading to the following result...

Theorem:
If I_1, I_2, I_3 are arbitrary integers, the Laurent expansions of the Gauss hypergeometric functions

$$
\begin{aligned}
& {}_2F_1(I_1 + a\varepsilon, I_2 + b\varepsilon; I_3 + \tfrac{p}{q} + c\varepsilon; z) \; , \\
& {}_2F_1(I_1 + \tfrac{p}{q} + a\varepsilon, I_2 + \tfrac{p}{q} + b\varepsilon; I_3 + \tfrac{p}{q} + c\varepsilon; z) \; , \\
& {}_2F_1(I_1 + \tfrac{p}{q} + a\varepsilon, I_2 + b\varepsilon; I_3 + c\varepsilon; z) \; , \\
& {}_2F_1(I_1 + \tfrac{p}{q} + a\varepsilon, I_2 + b\varepsilon; I_3 + \tfrac{p}{q} + c\varepsilon; z)
\end{aligned}
$$

are expressible in terms of multiple polylogarithms of arguments being powers of q-roots of unity and a new variable, that is an algebraic function of z, with coefficients that are ratios of polynomials.

Generalized Hypergeometric Function Example

$$F\left(\begin{array}{c} \{1+a_i\varepsilon\}^P \\ \frac{3}{2}+f\varepsilon,\ \{1+e_i\varepsilon\}^{P-2} \end{array}\middle| z\right) = \frac{(1+2f\varepsilon)}{2z}\frac{1-y}{1+y}\left\{\ln y + \sum_{k=1}^{\infty}\varepsilon^k \tilde{\tilde{\Phi}}_{k+1}(y)\right\},$$

$$F\left(\begin{array}{c} \{1+a_i\varepsilon\}^{R+2},\ \{2+d_i\varepsilon\}^{P-2-R} \\ \frac{3}{2}+f\varepsilon,\ \{1+e_i\varepsilon\}^{R}, \{2+c_i\varepsilon\}^{P-2-R} \end{array}\middle| z\right) = \frac{(1+2f\varepsilon)}{2z}\prod_{s=1}^{P-2-R}\frac{(1+c_s\varepsilon)}{(1+d_s\varepsilon)}$$

$$\left\{\frac{1-y}{1+y}\left[\ln y + \sum_{k=1}^{\infty}\varepsilon^k \Phi_{k+1}(y)\right] + \sum_{k=1}^{\infty}\varepsilon^k \tilde{\tilde{\Phi}}_{k+1}(y)\right\},$$

$$F\left(\begin{array}{c} \{1+a_i\varepsilon\}^{K+3},\ \{2+d_i\varepsilon\}^{P-3-K} \\ \frac{3}{2}+f\varepsilon,\ \{1+e_i\varepsilon\}^{K-L}, \{2+c_i\varepsilon\}^{P-2-K+L} \end{array}\middle| z\right) = \frac{(1+2f\varepsilon)}{2z}\frac{\prod_{s=1}^{P-2-K+L}(1+c_s\varepsilon)}{\prod_{s=1}^{P-3-K}(1+d_s\varepsilon)}\sum_{k=0}^{\infty}\varepsilon^k \tilde{\Phi}_{L+1+k}(y),$$

where F is the hypergeometric function ${}_PF_{P-1}$, R, K and L are integers with $0 \le R \le P-2$, $0 \le K \le P-3$, $0 \le L \le K$, the superscripts R and $K-L$ indicate the lengths of the parameter lists, $y = \frac{1-\sqrt{\frac{z}{z-1}}}{1+\sqrt{\frac{z}{z-1}}}$, and $\Phi_k(y)$ and $\tilde{\Phi}_k(y)$ are linear combinations of multiple polylogarithms of the square root of unity of weight $\mathbf{k}$.

Epsilon Expansion for Multiple Series

The structure of the expansion of ${}_PF_{P-1}$ leads to the structure for multiple series...
Let us define the linear combination of $S_a(2j-1)\cdots S_b(2j-1)$ as

$$\exp\left(\sum_{k=1}^{\infty}\frac{\varepsilon^k}{k}\bar{S}_k\right) = 1+\varepsilon\bar{S}_1+\varepsilon^2\left[\bar{S}_2+\bar{S}_1^2\right]$$

$$+\varepsilon^3\left[\bar{S}_1{}^3+3\bar{S}_1\bar{S}_2+2\bar{S}_3\right]$$

$$+\varepsilon^4\left[\bar{S}_1{}^4+6\bar{S}_1^2\bar{S}_2+3\bar{S}_2^2+8\bar{S}_1\bar{S}_3+6\bar{S}_4\right]+\mathcal{O}(\varepsilon^5)$$

$$= 1+\sum_{j=1}^{\infty}\varepsilon^j\Lambda_j\ ,$$

where $\bar{S}_k = S_k(2j-1)$ and all rational factors arising from the series expansion of the exponent are equal to 1.

Epsilon Expansion for Multiple Series

Let us consider the following sums:

$$\Sigma_{a_1,\cdots,a_k;b_1,\cdots,b_r;a;z} = \sum_{j=1}^{\infty} \frac{z^j}{j^a} S_{a_1} \cdots S_{a_k} \Lambda_{a_1} \cdots \Lambda_{b_r} .$$

Then:

$$\Sigma_{a_1,\cdots,a_k;b_1,\cdots,b_r;a;z} =\Big|_{u=-\frac{(1-y)^2}{y}}$$

$$= \frac{1-y}{1+y} \sum_{p,\vec{s}} c_{p,\vec{s}} \ln^p y \mathrm{Li}_{\binom{\vec{\sigma}}{\vec{s}}}(y)$$

$$\Sigma_{a_1,\cdots,a_k;b_1,\cdots,b_r;a;z} =\Big|_{u=-\frac{(1-y)^2}{y}}$$

$$= \sum_{p,\vec{s}} \tilde{c}_{p,\vec{s}} \ln^p y \mathrm{Li}_{\binom{\vec{\sigma}}{\vec{s}}}(y) \ , \quad c \geq 2 \ ,$$

where c is a positive integer, $c_{p,\vec{s}}$ and $\tilde{c}_{p,\vec{s}}$ are rational coefficients, $S_a(j-1) = \sum_{i=1}^{j-1} \frac{1}{i^a}$, is a harmonic sum and $\mathrm{Li}_{\binom{\vec{\sigma}}{\vec{s}}}(z)$ is the multiple polylogarithm of a square root of unity.

Note:

This result can also be extended to binomial sums and rational values of parameters.

Summary

- Feynman integrals may be expressed as linear combinations of Horn-type hypergeometric functions. Differential reduction can be applied to Horn-type hypergeometric functions with an arbitrary values of parameters.
- The number of master integrals required in differential reduction should agree with the number found using integration-by-parts techniques. This is an example of a “universal” type of result that we expect to arise from the existance of a hypergeometric representation.
- Dimensional regularization requires expansions of the hypergeometric functions about rational values of the parameters. The functions generated by the ε expansion can be specified via
  - a series representation,
  - iterative solution of differential equations;
  - the integral representation for hypergeometric functions;
- In particular, MUW-type algorithms do not suffice for all diagrams of interest. Multiple polylogarithms are not enough for all diagrams of interest.
- Generating functions in combination with direct solution of differential equations for the hypergeometric functions allow us to construct ε expansion for a large class of functions and analytically evaluate a large class of multiple series.

Appendix: References

- M.Yu. Kalmykov, JHEP 0604 (2006) 056 [hep-th/0602028]
- M.Yu. Kalmykov, B.F.L. Ward, S. Yost, JHEP 0702 (2007) 040 [hep-th/0612240]
- M.Yu. Kalmykov, B.F.L. Ward, S. Yost, JHEP 0710 (2007) 048 [arXiv:0707.3654]
- S.A. Yost, M.Yu. Kalmykov, B.F.L. Ward, in *Proceedings of the 34th International Conference in High Energy Physics* (ICHEP08), Philadelphia, 2008, eConf C080730 [arXiv:0808.2605]
- M.Yu. Kalmykov, B.F.L. Ward, S. Yost, JHEP 0711 (2007) 009 [arXiv:0708.0803]
- M.Yu. Kalmykov, V.V. Bytev, B.A. Kniehl, B.F.L. Ward, S.A. Yost, PoS(ACAT08) (2008) 125, [arXiv:0901.4716]
- V.V. Bytev, M.Yu. Kalmykov, B.A. Kniehl, B.F.L. Ward, S.A. Yost, in *Proceedings of the ILC 2009 Workshop*, Chicago (2009), [arXiv:0902.1352]
- M.Yu. Kalmykov, B.A. Kniehl, Nucl. Phys. B 809 (2009) 365 [arXiv:0807.0567]
- V.V. Bytev, M.Yu. Kalmykov, and B.A. Kniehl, Nucl. Phys. B 836 [FS] (2010) 129-170 [arXiv:0904.0214] [HYPERDIRE package]
- V.V. Bytev, M.Yu. Kalmykov, B.A. Kniehl, B.F.L. Ward, and S.A. Yost, in *Proceedings of the 35th International Conference in High Energy Physics*, Paris, 2010, PoS (ICHEP 2010) 135

Appendix: Shuffle Algebra

Let $\mathbb{X}$ be a finite set of letters and let $\mathbb{X}^\star$ denote the set of words on $\mathbb{X}$. The shuffle product is defined recursively by:

$$e ⧢ w = w ⧢ e = w \ ,$$
$$xu ⧢ yv = x(u ⧢ yv) + y(xu ⧢ v) \ ,$$

where $x, y \in \mathbb{X}$, $w, u, v \in \mathbb{X}^\star$, and e is an empty word.

The polylogarithm satisfies

$$\mathrm{Li}_u\,(z)\,\mathrm{Li}_v\,(z) = \mathrm{Li}_{u ⧢ v}\,(z) \ .$$

The case $z_1, \ldots z_r = 1$ gives a relation between multiple zeta values:

$$\zeta(u)\zeta(v) = \zeta(u ⧢ v) \ .$$

Multiple polylogarithms satisfy both shuffle and stuffle relations (next slide).

Appendix: Stuffle Algebra

The stuffle relations are based on a modification of the shuffle algebra.

Let $\mathbb{X}$ denote an alphabet and let $\mathbb{X}^*$ denote the set of words on $\mathbb{X}$.
The **stuffle product** $\star$ is defined recursively by

$$
\begin{aligned}
& e \star w = w \star e = w \ , \\
& x_1 w_1 \star x_2 w_2 = x_1 (w_1 \star x_2 w_2) + x_2 (x_1 w_1 \star w_2) + (x_1 \cdot x_2)(w_1 \star w_2) \ ,
\end{aligned}
$$

where $w, w_1, w_2 \in \mathbb{X}^\star$, $x_1, x_2 \in \mathbb{X}$ and $x_1 \cdot x_2$ is an associative product defined on the letters. The stuffle algebra is a graded Hopf algebra.

For multiple polylogarithms, the "letters" have the form $\mathrm{Li}_m\,(x)$, and the associative product of two letters $\mathrm{Li}_{m_1}\,(x_1)$ and $\mathrm{Li}_{m_2}\,(x_2)$ gives $\mathrm{Li}_{m_1+m_2}\,(x_1 x_2)$. The "words" have the form $w = \mathrm{Li}_{n_1,\ldots,n_r}\,(y_1, \ldots, y_r)$, and adding a letter to a word gives $x_1 w = \mathrm{Li}_{m_1,n_1,\ldots,n_r}\,(x_1, y_1, \ldots, y_r)$.

The case $x_1 = \cdots x_r = 1$, gives a relation between multiple zeta values:

$$\zeta(w_1)\zeta(w_2) = \zeta(w_1 \star w_2).$$