Exact results on $e^+e^- \to e^+e^- + 2\gamma$ at energies reached at the SLAC Linear Collider and CERN e^+e^- collider LEP

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We use the spinor methods of the CALKUL Collaboration, as realized by Xu, Zhang, and Chang, to calculate the differential cross section for $e^+e^- \to e^+e^- + 2\gamma$ for c.m. system energies in the regime of the SLAC Linear Collider (SLC) and CERN e^+e^- collider LEP. An explicit complete formula for the respective cross section is obtained. The leading-logarithmic approximation is used to check the formula. Applications of the formula to high-precision luminosity calculations at the SLC and/or LEP are discussed.

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I. INTRODUCTION

Currently, an unprecedented level of precision has been reached in both theory and experiment on the cross section σ_l of the basic luminosity process $e^+e^- \to e^+e^- + n\gamma$ for the SLAC Linear Collider (SLC) and CERN e^+e^- collider LEP, and this precision level has made possible the strongest tests to date of the $SU(2)_L \times U(1)$ theory in Z^0 physics [1]. The experimental precision is currently published [2] as 0.6%, whereas the theoretical precision as calculated by Jadach *et al.* [3] is currently published as 0.25% and is based on the Yennie-Frautschi-Suura (YFS) Monte Carlo event generator BHLUMI.[1] For a detailed illustration of how the error in the SLC and/or LEP luminosity enters the various standard model parameter measurements, see the paper by Dydak in Ref. [2]. Thus, 1% checks of the standard model in Z^0 physics are now in progress.

In the near term, the experimental error on σ_l at LEP is expected to improve to the 0.15% or better regime due to imminent hardware improvements [4]. Accordingly, it is important to improve the theoretical precision on σ_l in Ref. [3] to the 0.05% regime in the same near term, so that we will get our first glimpse of the 0.2% tests of the $SU(2)_L \times U(1)$ theory in Z^0 physics. Accordingly, the contributions to the error $\Delta\sigma_l$ in Table III of the third paper in Ref. [3] which violate our 0.05% requirement for the total error must be computed with an appropriately improved precision. The largest contribution to $\Delta\sigma_l$ in that table is due to the missing second-order bremsstrahlung contribution, which is itself 0.15%. Hence, in this paper, we compute the exact expression for the differential cross section for the respective process $e^+e^- \to e^+e^- + 2\gamma$ in the SLC and/or LEP energy regime, with the understanding that the e^+e^- scattering angles $\theta_{\bar{e},e}$ are always much larger than $2m_e/\sqrt{s}$.

Specifically, we shall employ methods originally pioneered by the CALKUL Collaboration in Refs. [5, 6]. We use these methods in the manner of Xu, Zhang, and Chang in Refs. [7]. Indeed, the CALKUL Collaboration computed the process $e^+e^- \to \gamma \to \mu^+\mu^- + 2\gamma$ in Ref. [6] using their original formulation of the spinor product method. In Refs. [8], Jadach *et al.* extended the CALKUL computation to include Z^0 exchange in $e^+e^- \to \mu^+\mu^- + 2\gamma$, thereby arriving at a formula of direct importance to high-precision Z^0 physics for the $e^+e^- \to Z^0 \to \mu^+\mu^- + n\gamma$ process. Entirely similar results, using the methods derived from those of Xu *et al.*, were obtained essentially simultaneously by Kleiss and van der Marck in Ref. [9]. Hence, what we present in this paper is the natural extension of the work of Refs. [6, 7, 9] to the double radiative Bhabha scattering process $e^+e^- \to e^+e^- + 2\gamma$.

We should emphasize that the process that we compute has applications in the Z^0 regime beyond just the luminosity. For, recently, there has been interest in wide angle events with two hard photons [10] at $\sqrt{s} \sim M_{Z^0}$ in $e^+e^- \to f\bar{f} + 2\gamma$, with $f = e, \mu$, etc. Here, we present results that are directly relevant to these events.

Our work is organized as follows. In Sec. II, we review the relevant spinor product notation and set our kinematic and notation conventions. In Sec. III, we analyze $e^+e^- \to e^+e^- + 2\gamma$ and present formulas for the complete scattering amplitude. In Sec. IV, we compare

[1]BHLUMI is available from the Computer Physics Communications program library.

our results for the differential cross section with known leading-logarithm approximations and thereby determine the size of the next-to-leading term in the second-order bremsstrahlung correction to the luminosity process at SLC and/or LEP. Section V contains some concluding remarks.

II. PRELIMINARIES

In order that our analysis be self-contained, we begin it in this section by stating our notation and conventions. This will also facilitate comparisons of our work with related efforts in the literature.

We first note that our metric will be that of Bjorken and Drell [11]. Our notation for Dirac γ matrices will follow that of Ref. [8] in the so-called chiral basis. Similarly, our conventions for the left- and right-handed couplings of the Z^0 to the electron are those of Ref. [8], so that

$$g_L = e\cot 2\theta_W, \qquad g_R = -e\tan\theta_W, \tag{1}$$

where e is the electric charge of the electron, so that it is negative. The rest mass of the Z^0 is denoted by M_Z,which we take [1] as 91.187 GeV. The Z^0 width will be denoted by Γ_Z and will be taken [1] as 2.492 GeV.

Our conventions for the metric, Dirac γ matrices and Z^0 charges we then complete with our spinor notation from Ref. [7]. Specifically, a massless spinor of four-momentum p and helicity λ is denoted by

$$\begin{aligned} |p,\lambda\rangle &\equiv u_\lambda(p) = v_{-\lambda}(p), \\ \langle p,\lambda| &\equiv \overline{u}_\lambda(p) = \overline{v}_{-\lambda}(p), \end{aligned} \tag{2}$$

with the normalization

$$\langle p,\lambda|\gamma^\mu|p,\lambda\rangle = 2p^\mu. \tag{3}$$

These spinors have a number of useful properties, a representative summary of which may be found in Ref. [7]. Here, we finalize our notational discussion of these objects by introducing the basic unit in which our amplitudes for our Bhabha scattering process will be expressed, namely, the spinor product. For two massless four-vectors p, q, we define the spinor product as

$$\langle p,q\rangle_\lambda = \langle p,-\lambda|q,\lambda\rangle\,. \tag{4}$$

It is sometimes convenient to introduce the triple product

$$\langle p,k,q\rangle_\lambda = \langle p,\lambda|\,\not{k}\,|q,\lambda\rangle\,, \tag{5}$$

where k is an arbitrary four-vector that need not be massless. This product is obviously linear in k, and when k is massless, it factorizes into a pair of spinor products:

$$\langle p,k,q\rangle_\lambda = \langle p,k\rangle_{-\lambda}\,\langle k,q\rangle_\lambda \quad \text{if } k^2=0. \tag{6}$$

Using the chiral basis for Dirac matrices in Ref. [8], the light-cone notation $p^\pm = p^0 \pm p^3$ with the three-axis directed along the incoming positron beam direction, and the complex perpendicular variable $p^\perp = p^1 + ip^2$, it can be shown that

$$|p,+\rangle = \frac{1}{\sqrt{p^+}}\begin{pmatrix} p^+ \\ p^\perp \\ 0 \\ 0 \end{pmatrix}, \qquad |p,-\rangle = \frac{1}{\sqrt{p^+}}\begin{pmatrix} 0 \\ 0 \\ -p^{\perp *} \\ p^+ \end{pmatrix}, \tag{7}$$

$$\begin{aligned} \langle p,+| &= \frac{-1}{\sqrt{p^+}}\left(0,0,p^+,p^{\perp *}\right), \\ \langle p,-| &= \frac{1}{\sqrt{p^+}}\left(p^\perp,-p^+,0,0\right) \end{aligned} \tag{8}$$

via standard Dirac equation manipulations. Under the crossing transformation, we need the analytic continuation of the square roots in (7) and (8) to negative values of p^+. We may use

$$|-p,\lambda\rangle = i\,|p,\lambda\rangle\,, \qquad \langle -p,\lambda| = i\,\langle p,\lambda|\,. \tag{9}$$

Evidently, we may now express our basic spinor products as explicit functions of their four-vector arguments. We get

$$\begin{aligned} \langle p,q\rangle_+ &= \frac{1}{\sqrt{p^+}\sqrt{q^+}}\left(p^\perp q^+ - p^+ q^\perp\right), \\ \langle p,q\rangle_- &= \operatorname{sgn} p^+ \ \operatorname{sgn} q^+\,\langle q,p\rangle_+^*\,, \end{aligned} \tag{10}$$

where the signs in the second expression are needed to account for the analytic continuation (9). For the triple product, we get

$$\begin{aligned} \langle p,k,q\rangle_+ = \frac{1}{\sqrt{p^+}\sqrt{q^+}}(&p^+k^-q^+ - p^+k^{\perp *}q^\perp \\ &-p^{\perp *}k^\perp q^+ + p^{\perp *}k^+q^\perp), \\ \langle p,k,q\rangle_- = \langle q,k,p\rangle_+\,. \end{aligned} \tag{11}$$

Finally, we note our convention for the photon polarizations, which we take from Ref. [7]. For a photon of helicity λ and four-momentum k, we define the polarization four-vector

$$\epsilon^\sigma(k,l,\lambda) = \lambda\frac{\langle l,-\lambda|\gamma^\sigma|k,-\lambda\rangle}{\sqrt{2}\,\langle l,-\lambda|k,\lambda\rangle}, \tag{12}$$

where l is a reference momentum such that $l^2 = k^2 = 0$.

This completes the notation needed for our analysis. We turn now to computation of the amplitudes of interest to us in the next section.

III. EXACT TWO-PHOTON BREMSSTRAHLUNG AMPLITUDES

In this section, we give exact expressions for all $O(\alpha^2)$ terms in the amplitude $e^+e^- \to e^+e^- + 2\gamma$, in the limit where the electron mass is negligible. The spinor conventions described in Sec. II allow these amplitudes to be expressed in a very compact form. The case of initial-state radiation in the s channel has already been obtained using these methods by Kleiss and van der Marck [9]. The complete s channel result, with muons in the final state, has also been obtained by Jadach *et al.* [8], but in a much less compact form.

The following kinematic conventions will be used. The momenta of the incoming and outgoing electron will be denoted p and q, while the corresponding helicities will be denoted λ and μ, respectively. The positron variables will be the same, but with a prime. The photon momenta and helicities will be denoted k_i and ρ_i, respectively.

The total amplitude may be written as a sum $\mathcal{M} = \mathcal{M}^s + \mathcal{M}^t$ of s- and t-channel contributions, with

$$\begin{aligned} \mathcal{M}^t &= \mathcal{M}_{pp} + \mathcal{M}_{ee} + \mathcal{M}_{pe}, \\ \mathcal{M}^s &= \mathcal{M}_{ii} + \mathcal{M}_{ff} + \mathcal{M}_{if}. \end{aligned} \tag{13}$$

Here, the subscripts p, e, i, and f indicate that a photon is emitted from the positron line, electron line, an initial-state line, and a final-state line, respectively. Helicity conservation plays an important role in simplifying the individual terms, as it is explained in Refs. [7, 9].

We begin with the t channel. Helicity conservation requires $\lambda = \mu$, $\lambda' = \mu'$ in all nonzero amplitudes. It is convenient to express these amplitudes in terms of helicity-dependent momentum variables

$$h_i = \begin{cases} p \\ q \end{cases} \text{if } \rho_i = \pm\lambda, \qquad h_i' = \begin{cases} p' \\ q' \end{cases} \text{if } \rho_i = \pm\lambda', \tag{14}$$

and to define the t variables

$$t = (p-q)^2, \qquad t' = (p'-q')^2, \qquad t_i = (p-q-k_i)^2. \tag{15}$$

When the photon helicities are identical, i.e., $\rho_1 = \rho_2 \equiv \rho$, the three terms in the t channel amplitude are given by

$$\mathcal{M}_{pp} = 4ie^4 G_{\lambda,-\lambda'}(t)\, \frac{\langle h, h'\rangle_\rho^2 \langle p', q'\rangle_\rho \langle p, q\rangle_{-\rho}}{\langle k_1, q', k_2\rangle_\rho \langle k_1, p', k_2\rangle_\rho}, \tag{16}$$

$$\mathcal{M}_{ee} = 4ie^4 G_{\lambda,-\lambda'}(t')\, \frac{\langle h, h'\rangle_\rho^2 \langle q, p\rangle_\rho \langle q', p'\rangle_{-\rho}}{\langle k_1, p, k_2\rangle_\rho \langle k_1, q, k_2\rangle_\rho}, \tag{17}$$

$$\mathcal{M}_{pe} = -4ie^4 \langle h, h'\rangle_\rho^2 \left\{ \frac{t_1 G_{\lambda,-\lambda'}(t_1)}{\langle p, k_1, q\rangle_\rho \langle p', k_2, q'\rangle_\rho} + (1 \leftrightarrow 2) \right\}. \tag{18}$$

The propagator factor for photon and Z^0 exchange is defined by

$$G_{\lambda,\mu}(z) = \frac{1}{z} + \frac{\left[(1-\lambda) - 4\sin^2\theta_W\right]\left[(1-\mu) - 4\sin^2\theta_W\right]}{4\sin^2 2\theta_W \left[z\,(1 + i\Gamma_Z/M_Z) - M_Z^2\right]}. \tag{19}$$

The Z^0 width Γ_Z is to be omitted in the t channel, but is present in the s channel expressions below.

The same amplitudes for opposite photon helicities are given by the more complicated expressions

$$\begin{aligned} \mathcal{M}_{pp} = \frac{4ie^4 \delta_{\rho_j,\lambda'}\, G_{\lambda,-\lambda'}(t)}{\langle k_i, q', k_j\rangle_{\lambda'} \langle k_i, p', k_j\rangle_{\lambda'}} \{ & \langle h_i, (q'+k_j), p'\rangle_{\lambda'} \langle q', (p'-k_i), h_j\rangle_{\lambda'} \\ & + \widehat{\Delta}'^{-1} \langle q', k_j\rangle_{-\lambda'} \langle p', h_j\rangle_{\lambda'} \langle h_i, (q'+k_j), k_i\rangle_{\lambda'} \langle k_i, p', k_j\rangle_{\lambda'} \\ & + \Delta'^{-1} \langle p', k_i\rangle_{\lambda'} \langle q', h_i\rangle_{-\lambda'} \langle k_j, (p'-k_i), h_j\rangle_{\lambda'} \langle k_i, q', k_j\rangle_{\lambda'} \}, \end{aligned} \tag{20}$$

$$\begin{aligned} \mathcal{M}_{ee} = \frac{4ie^4 \delta_{\rho_i,\lambda}\, G_{\lambda,-\lambda'}(t')}{\langle k_j, p, k_i\rangle_{\lambda} \langle k_j, q, k_i\rangle_{\lambda}} \{ & \langle h_j', (q+k_i), p\rangle_{\lambda} \langle q, (p-k_j), h_i'\rangle_{\lambda} \\ & + \Delta^{-1} \langle p, k_j\rangle_{\lambda} \langle q, h_j'\rangle_{-\lambda} \langle k_i, (p-k_j), h_i'\rangle_{\lambda} \langle k_j, q, k_i\rangle_{\lambda} \\ & + \widehat{\Delta}^{-1} \langle p, h_i'\rangle_{\lambda} \langle q, k_i\rangle_{-\lambda} \langle h_j', (q+k_i), k_j\rangle_{\lambda} \langle k_j, p, k_i\rangle_{\lambda} \}, \end{aligned} \tag{21}$$

$$\mathcal{M}_{pe} = 4ie^4 \left\{ G_{\lambda,-\lambda'}(t_1) \frac{\langle h_2', (\lambda h_2 + \rho_1 k_1), h_1\rangle_{\rho_1}^2}{\langle p, k_1, q\rangle_{\rho_1} \langle p', k_2, q'\rangle_{\rho_2}} + (1 \leftrightarrow 2) \right\}. \tag{22}$$

The indices $(i,j) = (1,2)$ or $(2,1)$ are chosen so that (20)–(21) are nonzero, and the denominators are defined by

$$\begin{aligned} \Delta &= (p - k_1 - k_2)^2, & \Delta' &= (p' - k_1 - k_2)^2, \\ \widehat{\Delta} &= (q + k_1 + k_2)^2, & \widehat{\Delta}' &= (q' + k_1 + k_2)^2. \end{aligned} \tag{23}$$

The s channel results are analogous. In this case, helicity conservation requires $\lambda' = -\lambda$ and $\mu' = -\mu$ for nonzero amplitudes. We define the helicity-dependent momentum variables

$$l_i = \begin{cases} p \\ p' \end{cases} \text{if } \rho_i = \pm\lambda, \qquad \widehat{l}_i = \begin{cases} q \\ q' \end{cases} \text{if } \rho_i = \mp\mu, \tag{24}$$

and the s variables

$$s=(p+p')^2, \qquad \widehat{s}=(q+q')^2, \qquad s_i=(p+p'-k_i)^2. \tag{25}$$

The three terms in the s channel amplitude are given by

$$\mathcal{M}_{ff}=4ie^4\lambda\mu G_{\lambda,\mu}(s)\,\frac{\left\langle l,\widehat{l}\right\rangle_\rho^2 \langle q,q'\rangle_\rho \langle p,p'\rangle_{-\rho}}{\langle k_1,q,k_2\rangle_\rho \langle k_1,q',k_2\rangle_\rho}, \tag{26}$$

$$\mathcal{M}_{ii}=4ie^4\lambda\mu G_{\lambda,\mu}(\widehat{s})\,\frac{\left\langle l,\widehat{l}\right\rangle_\rho^2 \langle p,p'\rangle_\rho \langle q,q'\rangle_{-\rho}}{\langle k_1,p,k_2\rangle_\rho \langle k_1,p',k_2\rangle_\rho}, \tag{27}$$

$$\mathcal{M}_{if}=4ie^4\lambda\mu \left\langle l,\widehat{l}\right\rangle_\rho^2 \left\{\frac{s_1 G_{\lambda,\mu}(s_1)}{\langle p,k_1,p'\rangle_\rho \langle q,k_2,q'\rangle_\rho}+(1\leftrightarrow 2)\right\} \tag{28}$$

for equal photon helicities, and by

$$\mathcal{M}_{ff}=\frac{-4ie^4\delta_{\rho_i,\mu}\,G_{\lambda,\mu}(s)}{\langle k_j,q,k_i\rangle_\mu \langle k_j,q',k_i\rangle_\mu}\Big\{ \langle q,(q'+k_j),l_i\rangle_\mu \langle l_j,(q+k_i),q'\rangle_\mu$$
$$+\widehat{\Delta}'^{-1}\langle q',k_j\rangle_\mu \langle q,l_j\rangle_{-\mu} \langle k_i,(q'+k_j),l_i\rangle_\mu \langle k_j,q,k_i\rangle_\mu$$
$$+\widehat{\Delta}^{-1}\langle q,k_i\rangle_{-\mu} \langle q',l_i\rangle_\mu \langle l_j,(q+k_i),k_j\rangle_\mu \langle k_j,q',k_i\rangle_\mu\Big\}, \tag{29}$$

$$\mathcal{M}_{ii}=\frac{-4ie^4\delta_{\rho_i,\lambda}\,G_{\lambda,\mu}(\widehat{s})}{\langle k_j,p,k_i\rangle_\lambda \langle k_j,p',k_i\rangle_\lambda}\Big\{ \left\langle \widehat{l}_j,(p'-k_i),p\right\rangle_\lambda \left\langle p',(p-k_j),\widehat{l}_i\right\rangle_\lambda$$
$$+\Delta^{-1}\langle p,k_j\rangle_\lambda \left\langle p',\widehat{l}_j\right\rangle_{-\lambda} \left\langle k_i,(p-k_j),\widehat{l}_i\right\rangle_\lambda \langle k_j,p',k_i\rangle_\lambda$$
$$+\Delta'^{-1}\left\langle p,\widehat{l}_i\right\rangle_\lambda \langle p',k_i\rangle_{-\lambda} \left\langle \widehat{l}_j,(p'-k_i),k_j\right\rangle_\lambda \langle k_j,p,k_i\rangle_\lambda\Big\},$$

$$\mathcal{M}_{if}=4ie^4\lambda\mu\left\{G_{\lambda,\mu}(s_1)\frac{\left\langle \widehat{l}_2,(l_2-k_1),l_1\right\rangle_{\rho_1}^2}{\langle p,k_1,p'\rangle_{\rho_1}\langle q,k_2,q'\rangle_{\rho_2}}+(1\leftrightarrow 2)\right\} \tag{30}$$

for opposite photon helicities.

The amplitudes $\mathcal{M}$ above are related by crossing symmetries. Interchanging the incoming positron and outgoing electron, so that

$$p'\leftrightarrow -q, \qquad \lambda'\leftrightarrow -\mu, \tag{31}$$

interchanges the s and t channel amplitudes

$$\mathcal{M}_{pp}\leftrightarrow \mathcal{M}_{ff}, \qquad \mathcal{M}_{ee}\leftrightarrow \mathcal{M}_{ii}, \qquad \mathcal{M}_{pe}\leftrightarrow \mathcal{M}_{if}, \tag{32}$$

while interchanging the incoming electron and outgoing positron, so that

$$p\leftrightarrow -q', \qquad \lambda\leftrightarrow -\mu' \tag{33}$$

gives

$$\mathcal{M}_{pp}\leftrightarrow \mathcal{M}_{ii}, \qquad \mathcal{M}_{ee}\leftrightarrow \mathcal{M}_{ff}, \qquad \mathcal{M}_{pe}\leftrightarrow \mathcal{M}_{if}. \tag{34}$$

These expressions are useful in practice, since, together, they allow all of the amplitudes to be obtained from only four expressions; for example, (16), (18), (20), and (22). The form of the expression depends only on whether the photon helicities are equal or opposite, and whether they are emitted from the same or different fermion lines.

This completes our derivation of the exact second-order matrix element for the process $e^+e^-\to e^+e^-+2\gamma$. We turn now to some of its applications. This we do in the next section.

IV. SAMPLE RESULTS

In this section we illustrate our exact result in the context of its main purpose, which is to check the nonleading bremsstrahlung correction for two hard photons in the low angle regime of Bhabha scattering in our SLC and/or LEP luminosity calculations in Ref. [3]. Thus, we will compare our exact results with the leading log expectations in the low angle Bhabha scattering luminosity regime. We begin with a discussion of the differential cross section associated with our exact result.

Using entirely standard manipulations [11], we get the following expression for the exact differential cross section for $e^+e^-\to e^+e^-+2\gamma$ in the Z^0 resonance region:

$$d\sigma=\frac{1}{2!}\delta^4(p+p'-q-q'-k_1-k_2)\times\frac{\left|\overline{\mathcal{M}}\right|^2}{(2\pi)^8 s}\frac{d^3\mathbf{q}\,d^3\mathbf{q}'\,d^3\mathbf{k_1}\,d^3\mathbf{k_2}}{2^4q^0q'^0k_1^0k_2^0}, \tag{35}$$

where the averaged and summed squared matrix element may be expressed as a sum over helicities,

$$|\overline{\mathcal{M}}|^2 \equiv \frac{1}{4} \sum_{\lambda,\lambda',\mu,\mu',\rho_1,\rho_2=\pm 1} |\mathcal{M}(\lambda,\lambda',\mu,\mu',\rho_1,\rho_2)|^2 , \tag{36}$$

using the total exact amplitude derived in the previous section. We will compare the exact cross section (35) with the leading-log expectations. This comparison requires a leading-log approximation to the exact cross section, namely

$$d\sigma_{LL} = \frac{1}{2!}\delta^4(p+p'-q-q'-k_1-k_2) \times \frac{|\overline{\mathcal{M}}|^2_{LL}}{(2\pi)^8 s} \frac{d^3\mathbf{q}\, d^3\mathbf{q}'\, d^3\mathbf{k_1}\, d^3\mathbf{k_2}}{2^4 q^0 q'^0 k_1^0 k_2^0}, \tag{37}$$

where the leading-log summed squared matrix element is given by (38) below. It is to the derivation of the latter equation that we now turn.

For our leading-log representation of $|\overline{\mathcal{M}}|^2_{LL}$, we follow the development given by Jadach and Ward in Ref. [12]. Specifically, we generalize the initial-state s-channel double bremsstrahlung leading-log differential cross section in Ref. [12] to include analogous contributions from the final state and all other channels containing collinear singularities. This leads to an expression for the leading-log version of the squared matrix element (36) given as a sum of six similar terms, labeled by the fermion lines emitting the photons (initial, final, positron, electron, or mixed):

$$|\overline{\mathcal{M}}|^2_{LL} \equiv |\mathcal{L}_i + \mathcal{L}_f + \mathcal{L}_p + \mathcal{L}_e - \mathcal{L}_m - \mathcal{L}_{m'}| . \tag{38}$$

The individual terms in (38) are given by

$$\begin{aligned}
\mathcal{L}_i &= \frac{2e^8}{s\widehat{s}D(p,p')}\left[f(p,p')\left|\overline{\mathcal{M}}_B(\widehat{s},t_i')\right|^2 + f(p',p)\left|\overline{\mathcal{M}}_B(\widehat{s},t_i)\right|^2\right],\\
\mathcal{L}_f &= \frac{2e^8}{s\widehat{s}D(q,q')}\left[f(-q',-q)\left|\overline{\mathcal{M}}_B(s,t_f)\right|^2 + f(-q,-q')\left|\overline{\mathcal{M}}_B(s,t_f')\right|^2\right],\\
\mathcal{L}_p &= \frac{2e^8}{tt'D(p',q')}\left[f(-q',p')\left|\overline{\mathcal{M}}_B(s_p,t)\right|^2 + f(p',-q')\left|\overline{\mathcal{M}}_B(\widehat{s}_p,t)\right|^2\right],\\
\mathcal{L}_e &= \frac{2e^8}{tt'D(p,q)}\left[f(p,-q)\left|\overline{\mathcal{M}}_B(\widehat{s}_e,t')\right|^2 + f(-q,p)\left|\overline{\mathcal{M}}_B(s_e,t')\right|^2\right],\\
\mathcal{L}_m &= \frac{2e^8}{uu'D(p,q')}\left[f(-q',p)\left|\overline{\mathcal{M}}_B(s_m,t_m)\right|^2 + f(p,-q')\left|\overline{\mathcal{M}}_B(\widehat{s}_m,t_m')\right|^2\right],\\
\mathcal{L}_{m'} &= \frac{2e^8}{uu'D(p',q)}\left[f(p',-q)\left|\overline{\mathcal{M}}_B(\widehat{s}_{m'},t_{m'})\right|^2 + f(-q,p')\left|\overline{\mathcal{M}}_B(s_{m'},t_{m'}')\right|^2\right]
\end{aligned} \tag{39}$$

where the notation is defined as follows. The basic s,t,u invariants are defined by (15), (25) and

$$u=(p-q')^2, \qquad u'=(p'-q)^2. \tag{40}$$

The pure Born amplitude with no photons is $\mathcal{M}_B(s_B,t_B)$, where the effective Born parameters s_B, t_B are as shown, with

$$t_i = \frac{-\widehat{s}t}{t+u}, \qquad t_i' = \frac{-\widehat{s}t'}{t'+u'}, \qquad t_f = \frac{-st}{t+u'}, \qquad t_f' = \frac{-st'}{t'+u}, \tag{41}$$

$$s_p = \frac{-st}{s+u'}, \qquad \widehat{s}_p = \frac{-\widehat{s}t}{\widehat{s}+u}, \qquad s_e = \frac{-st'}{s+u}, \qquad \widehat{s}_e = \frac{-\widehat{s}t'}{\widehat{s}+u'}, \tag{42}$$

$$s_m = \frac{-su'}{s+t}, \qquad \widehat{s}_m = \frac{-\widehat{s}u'}{\widehat{s}+t'}, \qquad t_m = \frac{-tu'}{s+t}, \qquad t_m' = \frac{-t'u'}{\widehat{s}+t'}, \tag{43}$$

$$s_{m'} = \frac{-su}{s+t'}, \qquad \widehat{s}_{m'} = \frac{-\widehat{s}u}{\widehat{s}+t}, \qquad t_{m'} = \frac{-tu}{\widehat{s}+t}, \qquad t_{m'}' = \frac{-t'u}{s+t'}. \tag{44}$$

The denominator factor $D(p_1,p_2)$ is defined to be

$$D(p_1,p_2) = (p_1\cdot k_1)(p_1\cdot k_2)(p_2\cdot k_1)(p_2\cdot k_2)(p_1\cdot p_2)^{-4}, \tag{45}$$

and if the indices $(i,j)=(1,2)$ or $(2,1)$ are chosen so that $p_1\cdot k_i + p_2\cdot k_i \ge p_1\cdot k_j + p_2\cdot k_j$, the form factors $f(p_1,p_2)$ are given by

$$f(p_1,p_2) = Y\left(\frac{p_1\cdot k_i}{p_1\cdot p_2}, \frac{p_1\cdot k_j}{p_1\cdot(p_2-k_i)}, \frac{p_2\cdot k_j}{p_2\cdot(p_1-k_i)}\right) + Y\left(\frac{p_1\cdot k_i}{p_1\cdot(p_2-k_j)}, \frac{p_1\cdot k_j}{p_1\cdot p_2}, \frac{p_2\cdot k_j}{p_1\cdot p_2}\right), \tag{46}$$

where $Y(x,u,v) = (1-x)^2\left[(1-u)^2 + (1-v)^2\right]$.

We should note that, unlike the original leading-logarthmic (LL) expression in Ref. [12], the result (37) does not control the next-to-leading-log corrections. We will now discuss the comparison of exact and leading-log differential cross sections (35) and (37), respectively, in the SLC and/or LEP luminosity regime.

We have a made a detailed comparison of the exact and leading-log versions of the cross section (35) in the luminosity regime. We illustrate these comparisons in Figs. 1 and 2. In Fig. 1, the two photons are emitted near

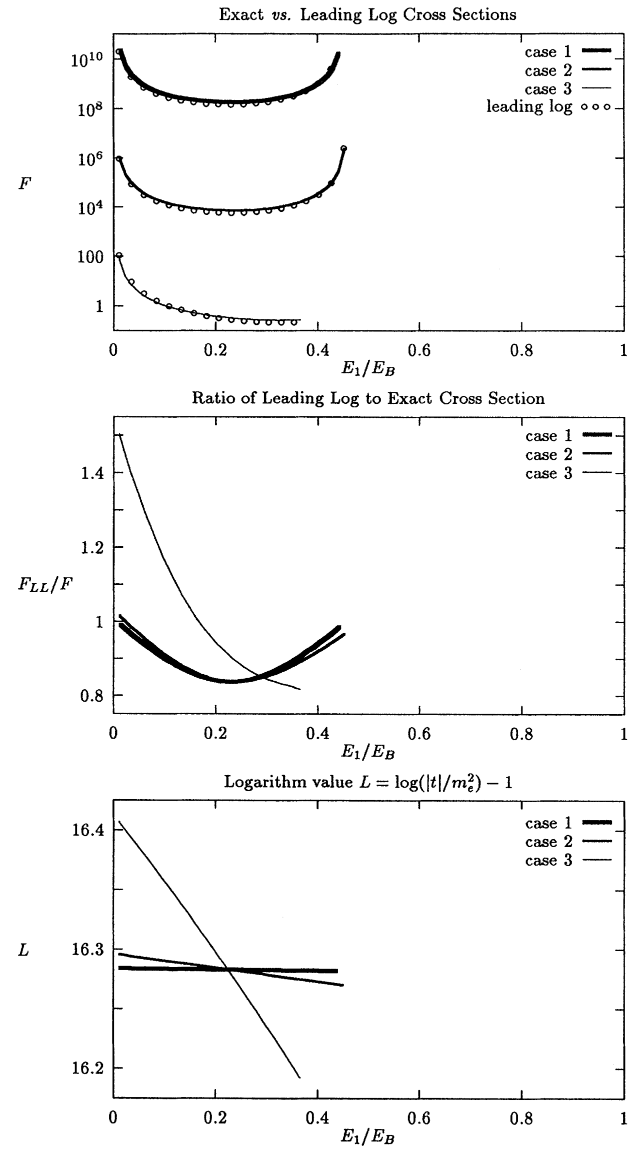

FIG. 1. Comparison of LL and exact results for low angle Bhabha scattering in the SLC and/or LEP luminosity regime. The functions F are the normalized differential cross sections defined in Ref. [8]. We parametrize eight-dimensional $e^+e^-\gamma\gamma$ phase space in terms of seven angles and the energy of one-photon E_1. We plot the dependence on photon energy E_1 for fixed values of angles which are listed in Table I. Here, in the table, the subscripts 1 and 2 refer to the photons and where the photon angles are relative to the incoming positron direction, so that, in this figure, both photons are emitted along the incoming positron direction. Note that the energy of the outgoing charged particles is required to remain above 0.5 of the beam energy E_B. For the three cases 1, 2, 3 of kinematics shown in the table, the plots show respectively the individual exact and LL cross sections, their ratio. The cases 1 and 2 are relevant to NLL correction, while case 3 covers the NNLL region of the phase space. The size of the dominant big logarithm $\ln(|t|/{m_e}^2) - 1$, which determines the probability to radiate in the process, is also plotted.

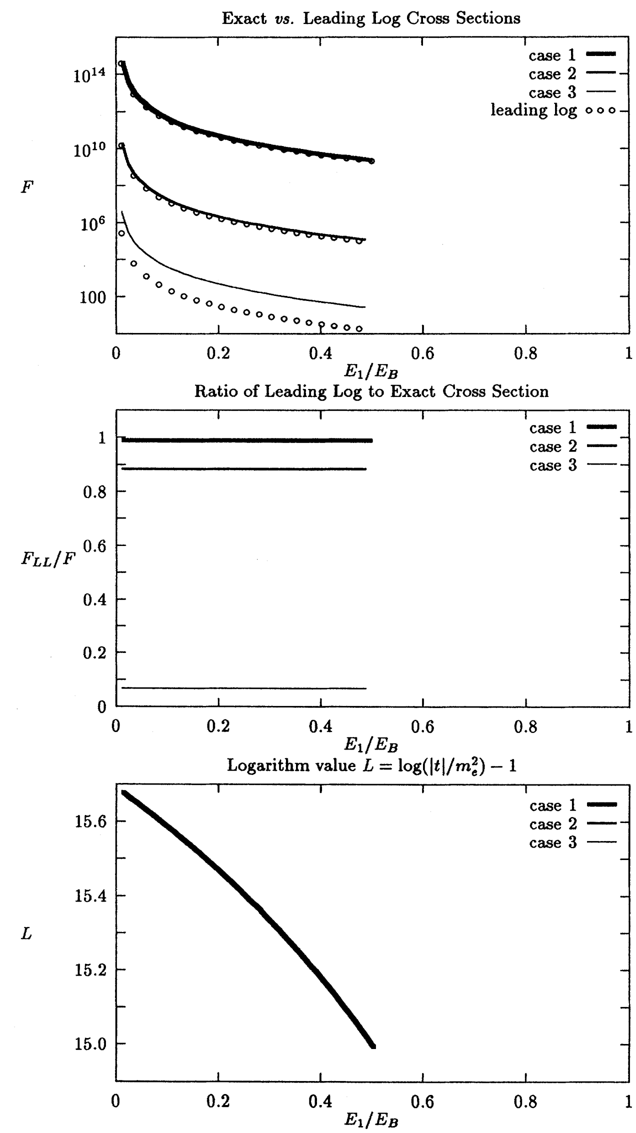

FIG. 2. The same plots as those given in Fig. 1 with photon 1's spherical angles measured relative to the incoming positron direction and with photon 2's angles measured relative to the incoming electron direction. Thus, the two photons are emitted in generally opposite directions. The specific kinematic input is summarized in Table II. The cut on the charged particles' outgoing energies is the same as in Fig. 1.

TABLE I. Kinematic variables for Fig. 1.

Case	θ_1	ϕ_1	θ_2	ϕ_2	θ_{e+}	ϕ_{e+}	θ_{e-}
1	6.4×10^{-3} °	0°	6.4×10^{-3} °	180°	4.9°	0°	2.7°
2	8×10^{-2} °	0°	8×10^{-2} °	180°	4.9°	0°	2.7°
3	1°	0°	1°	180°	4.9°	0°	2.7°

the incoming e^+ line in direction, with angles as given in the figure, whereas in Fig. 2, one photon is emitted along the direction of each incoming charged particle, with the respective angles as given in the figure. The final particles in the events associated with Figs. 1 and 2 are all taken to fall within the typical LEP-type trigger region as described in Ref. [3] and as we indicate in the figures. What we see from Figs. 1 and 2, cases 1 and 2, is that the LL result is always within 20% of the exact result in the relevant region of the phase space. We have verified that this is true throughout the wide range of the kinematic variables in the respective region of phase space. This indicates that the error estimate for the next-to-leading (NLL) $O(\alpha^2)$ double bremsstrahlung effects not included in BHLUMI 2.01 in Ref. [3] is rather conservative. Case 3 in Figs. 1 and 2 is relevant for next-to-next-to-leading (NNLL) $O(\alpha^2)$ corrections, which are at the level of 10^{-4} of the integrated cross section. The present calculation opens a path to implementation of the NLL $O(\alpha^2)$ double hard bremmsstrahlung effect into BHLUMI in order to move it closer to the desired 0.05% precision tag needed to support the 0.15% experimental errors expected in the near term high-precision Z^0 physics program at LEP. This implementation will appear elsewhere [13].

The last curves in the figures show the size of the big logarithm $L = \ln(|t|/m_e^2) - 1$ in the respective regions of the phase space. We see that L varies significantly and, hence, that giving it the proper argument $|t|$ instead of s in the low angle regime suppresses possible next-to-leading-log corrections. This then is consistent with the implementation of YFS (Yennie-Frautschi-Suura) exponentiation in BHLUMI, where two of us (S.J. and B.F.L.W.) have also found that the proper argument for L in the respective radiation probability is $|t|$.

We should also point out that our exact 2γ emission results in low angle Bhabha scattering are directly relevant to the QED expectations for high mass 2γ states in wide angle Bhabha scattering at Z^0 energies. Thus, recent interest [10] in such events warrants a detailed assessment of such phenomena using our results in this paper. This assessment will appear elsewhere [13]. Our main objective in this paper is to present the exact results for $e^+e^- \to e^+e^- + 2\gamma$ at order α^2 and to assess the error estimate of the respective missing part of the order α^2 bremsstrahlung effect in BHLUMI as it is given in Ref. [3].

V. CONCLUSIONS

In this paper we have used the methods of the CALKUL Collaboration to compute the important process of two-photon bremsstrahlung in Bhabha scattering in the Z^0 resonance region at $O(\alpha^2)$. We have compared our exact results for the corresponding differential cross section with leading-log expectations as a check of the results in Ref. [3] on the total precision of the Monte Carlo program BHLUMI2.01 [14].

Specifically, we have presented new formulas for the process $e^+e^- \to e^+e^- + 2\gamma$ at $O(\alpha^2)$ in the Z^0 resonance region. We have compared our exact results with the LL expectations for the respective differential cross sections and we find that the two differential cross sections are within 20% of each other throughout the relevant region of the phase space. This indicates that the estimate in Ref. [3] for the contribution of the missing NLL $O(\alpha^2)$ bremsstrahlung in BHLUMI2.01 [14] to its total precision of 0.25% is rather conservative. The implementation of this missing part of the $O(\alpha^2)$ bremsstrahlung into BHLUMI will be reported elsewhere [13].

We emphasize that our new formulas are valid both at low and wide electron, positron scattering angles as defined by the typical LEP and/or SLC trigger in the luminosity regime; i.e., low refers to scattering angles in the luminosity regime and wide refers to such angles larger than the trigger angles. This means that our results have an immediate application to the recent discussion of large two-photon mass wide angle lepton pair production events in Ref. [10]. Such applications will appear elsewhere [13].

In summary, the exact results, which we have presented for the two-photon bremsstrahlung process in Bhabha scattering in the Z^0 resonance region, allow us to make an important step in reducing the error on the theoretical prediction for the LEP and/or SLC luminosity process to below the 0.1% regime. We look forward to the

TABLE II. Kinematic variables for Fig. 2.

Case	θ_1	ϕ_1	θ_2	ϕ_2	θ_{e+}	ϕ_{e+}	θ_{e-}
1	6.4×10^{-3} °	0°	6.4×10^{-3} °	180°	2.7°	0°	2.7°
2	8×10^{-2} °	0°	8×10^{-2} °	180°	2.7°	0°	2.7°
3	1°	0°	1°	180°	2.7°	0°	2.7°

attendant refinement in the precision of the respective standard model tests in Z^0 physics.

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