All-Order ε-Expansion of Generalized Hypergeometric Functions

Scott Yost

PRINCETON UNIVERSITY

with M. Kalmykov, B.F.L. Ward

Representing Feynman Diagrams

- It would be very useful to have a general means of representing a Feynman diagram with an arbitrary number of loops and legs.
- Reduction techniques to represent a given diagram in terms of a class of more elementary integrals are very useful in computations.
- One of the most powerful representations of Feynman diagrams is in terms of **hypergeometric functions**.
- Since the diagrams typically diverge in 4 dimensions, an expansion must be developed in a small parameter about d=4: this is called an **ε-expansion**.

Generalized Hypergeometric Functions

- The generalized hypergeometric function ${}_pF_q$ has expansion

$$ {}_pF_q\left(\begin{matrix} a_1, & \ldots, & a_n \\ b_1, & \ldots, & b_n \end{matrix} \middle| z \right) = \sum_{j=0}^{\infty} \frac{\prod_{i=1}^{p} (a_i)_j}{\prod_{k=1}^{p} (b_k)_j} \frac{z^j}{j!} $$

  with $(a)_j = \Gamma(a+j)/\Gamma(a)$ the "Pochhammer symbol" and the b-parameters not to be negative integers.

- Special values of the parameters, in which the parameters a_i and b_i differ from given parameters A_i, B_i by a shifts proportional to a small parameter ε are useful in dimensional regularized diagrams, where the dimension of space-time is shifted to $d = 4 - 2\varepsilon$ to regulate UV and IR divergences.

Epsilon Expansions

The hypergeometric function can be expanded in powers of the parameter ε. The terms in this expansion multiply poles $1/\varepsilon^n$ from UV and IR divergences. Higher-order terms are needed in the expansion for higher-loop graphs.

Classes of functions known as multiple polylogarithms

$$\mathrm{Li}_{k_1,k_2,\ldots,k_n}\left(z_1, z_2, \ldots, z_n\right) = \sum_{m_1>m_2>\cdots>m_n>0} \frac{z_1^{m_1} z_2^{m_2} \cdots z_n^{m_n}}{m_1^{k_1} m_2^{k_2} \cdots m_n^{k_n}}$$

have proven useful for representing the coefficients of the ε-expansions of a large class of hypergeometric functions.

Integer Parameters

- In the case when the parameters A, B, C are integers, the ε-expansion may be written in terms of harmonic polylogarithms.

  [Remiddi and Vermaseren, Int. J. Mod. Phys. A15 (2000), 725]

- A harmonic polylogarithm of weight w is defined recursively in terms of a parameter-vector $\vec{m}_w$ of dimension w having entries $0, \pm 1$ only.

Harmonic Polylogarithms

- The definition of the harmonic polylogarithm is recursive, starting from, for $\overrightarrow{0}_w = (0,\ldots,0)$,

$$\mathrm{H}\left(\vec{0}_w; z\right) = \frac{1}{w!} \ln^w z$$

and for $\overrightarrow{m}_w = (a, m_{w\text{-}1}) \neq \overrightarrow{0}_{\mathrm{w}}$,

$$\mathrm{H}\left(\vec{m}_w; z\right) = \int_0^z dx \; f(a;x)\,\mathrm{H}\left(\vec{m}_{w-1}; x\right)$$

with

$$f(0;x) = \frac{1}{x}, \quad f(1,x) = \frac{1}{1-x}, \quad f(-1,x) = \frac{1}{1+x}$$

Multiple Polylogarithms

- Harmonic polylogarithms are a special case of multiple polylogarithms, which may be expanded as

$$\mathrm{Li}_{k_1,\cdots,k_n}(z) = \sum_{m_1>m_2>\cdots>m_n>0} \frac{z^{m_1}}{m_1^{k_1} m_2^{k_2} \cdots m_n^{k_n}}$$

or expressed as iterated Chen integrals over differential forms $\omega_0 = dz/z$ and $\omega_1 = dz/(1-z)$:

$$\mathrm{Li}_{k_1,\cdots,k_n}(z) = \int_0^z \omega_0^{k_1-1}\omega_1 \cdots \omega_0^{k_n-1}\omega_1$$

Generalization

- Our goal was to generalize this result to cases where the parameters could be nearly integers or half-integers. In this case, a new type of sum is generated:

$$\sum_{j=1}^{\infty} \frac{1}{\binom{2j}{j}^k} \frac{z^j}{j^c} S_{a_1}(j-1) \cdots S_{a_p}(j-1)\, S_{b_1}(2j-1) \cdots S_{b_p}(2j-1)$$

These are ⟨ **generalized harmonic** / **inverse binomial** / **binomial** ⟩ **sums** for $k = 0,\ 1,\ -1$

Gauss Hypergeometric Functions

- We will discuss a particular class of Gauss Hypergeometric functions, which have the form

$$ {}_2F_1(A+a\varepsilon, B+b\varepsilon; C+c\varepsilon; z) = {}_2F_1\left(\left.\begin{matrix} A+a\varepsilon, B+b\varepsilon \\ C+c\varepsilon \end{matrix}\right| z\right) $$

$$ = \sum_{j=0}^{\infty} \frac{(A+a\varepsilon)_j (B+b\varepsilon)_j}{(C+c\varepsilon)_j} \frac{z^j}{j!} $$

where in dimensional regularization, $d = 4 - 2\varepsilon$.

Theorem

- The ε-expansion of a Gauss hypergeometric function

$$ {}_2F_1(A+a\varepsilon, B+b\varepsilon; C+c\varepsilon; z) $$

with A, B, C integers or half-integers may be expressed in terms of harmonic polylogarithms with polynomial coefficients.

[M. Yu. Kalmykov, JHEP04(2006) 256, M.Yu. Kalmykov, B.F.L. Ward, S.A. Yost, JHEP02(2007) 040]

Reduction Algorithm

- The proof begins with the observation that any Gauss hypergeometric function can be written as a linear combination of two others with parameters differing from the original parameters by an integer.
- Specifically,

$$P(a,b,c,z)\,{}_2F_1\left(a+I_1, b+I_2; c+I_3; z\right) = \left\{Q_1(a,b,c,z)\frac{d}{dz} + Q(a,b,c,z)\right\}\,{}_2F_1\left(a,b;c;z\right)$$

with a, b, c arbitrary parameters, I_1, I_2, I_3 integers, and P, Q_1, Q_2 polynomials in the parameters and argument z.

Reduction Algorithm

- In this way, the given hypergeometric function can be reduced to a combination of five basis functions and their first derivatives:

$$ {}_2F_1\left(a, b\varepsilon; 1+c\varepsilon; z\right), \quad {}_2F_1\left(a\varepsilon, b\varepsilon; \tfrac{1}{2}+c\varepsilon; z\right), $$

$$ {}_2F_1\left(\tfrac{1}{2}+a\varepsilon, b\varepsilon; 1+c\varepsilon; z\right), \quad {}_2F_1\left(\tfrac{1}{2}+a\varepsilon, b\varepsilon; \tfrac{1}{2}+c\varepsilon; z\right), \quad {}_2F_1\left(\tfrac{1}{2}+a\varepsilon, \tfrac{1}{2}+b\varepsilon; \tfrac{1}{2}+c\varepsilon; z\right) $$

- In fact, it is known that only the first two are algebraically independent, so to prove the theorem, it is sufficient to consider only these two basis functions and show that they can be expressed as harmonic polylogarithms.

Outline of Proof

- The proof proceeds by writing a differential equation satisfied by the basis hypergeometric functions, and expanding the solution in powers of ε^n.
- The coefficients of these powers can then be constructed iteratively and recognized as harmonic polylogarithms.
- Obtaining the k^{th} coefficient requires knowledge of the previous ones, in this construction.

Follow-Up Results

More recently, Kalmykov and Kniehl have shown that the ε-expansions of Gauss hypergeometric functions with certain rational parameters can be expressed in terms of multiple polylogarithms with coefficiencts that are ratios of polynomials with complex coefficients. These functions have the form

$$ {}_2F_1\left(I_1+\tfrac{p}{q}+a\varepsilon, I_2+b\varepsilon, I_3+c\varepsilon; z\right), \qquad {}_2F_1\left(I_1+a\varepsilon, I_2+b\varepsilon, I_3+\tfrac{p}{q}+c\varepsilon; z\right), $$

$$ {}_2F_1\left(I_1+\tfrac{p}{q}+a\varepsilon, I_2+b\varepsilon, I_3+\tfrac{p}{q}+c\varepsilon; z\right), \qquad {}_2F_1\left(I_1+\tfrac{p}{q}+a\varepsilon, I_2+\tfrac{p}{q}+b\varepsilon, I_3+\tfrac{p}{q}+c\varepsilon; z\right) $$

with I_1, I_2, I_3, p, q integers.

[M. Yu. Kalmykov, B. A. Kniehl, arXiv:0807.0567 (hep-th)]

Outlook

- This is just a very brief introduction to hypergeometric function approach to Feynman diagrams.
- One goal would be to combine the results into a software package that could be used to simplify Feynman integrals using an algorithm based on this representation.
- Conversely, mathematicians have been using results motivated by Feynman diagrams to discover new relations among hypergeometric functions and related functions. This is a fertile area of interaction between mathematics and physics.