

# All-Order $\varepsilon$ -Expansion of Gauss Hypergeometric Functions with Integer and Half-Integer Values of Parameters



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# Representing Feynman Diagrams

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- It would be very useful to have a general means of representing a Feynman diagram with an arbitrary number of loops and legs.
- Reduction techniques to represent a given diagram in terms of a class of more elementary integrals are very useful in computations.
- Since the diagrams typically diverge in 4 dimensions, an expansion must be developed in a small parameter about  $d=4$ .

# Hypergeometric Function Approach

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- One of the most powerful representations of Feynman diagrams is in terms of **hypergeometric functions**.
- Much work has been done on finding the representation of various diagrams in terms of hypergeometric functions, and finding recursion relations among them which can be the basis for a reduction algorithm.

# Generalized Hypergeometric Functions

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- It has been known since the early 70's that Feynman diagrams may be expressed in terms of **generalized hypergeometric functions**.
- This representation has an advantage of efficiency – for example, the 4-point massive scalar box diagram may be expressed as 192 dilogs of – or a single hypergeometric function of several variables. This helps to cancel spurious singularities.

[D.S. Kershaw, Phys. Rev. D8 (1973) 2708]

# Generalized Hypergeometric Functions

- The generalized hypergeometric function  ${}_pF_q$  has expansion

$${}_pF_q \left( \begin{matrix} a_1, & \dots, & a_n \\ b_1, & \dots, & b_n \end{matrix} \middle| z \right) = \sum_{j=0}^{\infty} \frac{\prod_{i=1}^p (a_i)_j}{\prod_{k=1}^q (b_k)_j} \frac{z^j}{j!}$$

with  $(a)_j = \Gamma(a+j)/\Gamma(a)$  the “Pochhammer symbol” and the  $b$ -parameters not to be negative integers.

- Special values of the parameters, in which the parameters  $a_i$  and  $b_i$  differ from given parameters  $A_i, B_i$  by a shifts proportional to a small parameter  $\varepsilon$  are useful in dimensional regularized diagrams, where the dimension of space-time is shifted to  $d = 4 - 2\varepsilon$  to regulate UV and IR divergences.

# Epsilon Expansions

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The hypergeometric function can be expanded in powers of the parameter  $\epsilon$ . The terms in this expansion multiply poles  $1/\epsilon^n$  from UV and IR divergences. Higher-order terms are needed in the expansion for higher-loop graphs.

Classes of functions known as **multiple polylogarithms**

$$\text{Li}_{k_1, k_2, \dots, k_n}(z_1, z_2, \dots, z_n) = \sum_{m_1 > m_2 > \dots > m_n > 0} \frac{z_1^{m_1} z_2^{m_2} \dots z_n^{m_n}}{m_1^{k_1} m_2^{k_2} \dots m_n^{k_n}}$$

have proven useful for representing the coefficients of the  $\epsilon$ -expansions of a large class of hypergeometric functions.

# Integer Parameters

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- In the case when the parameters  $A, B, C$  are integers, the  $\varepsilon$ -expansion may be written in terms of harmonic polylogarithms.

[Remiddi and Vermaseren, Int. J. Mod. Phys. A15 (2000), 725]

- A harmonic polylogarithm of weight  $w$  is defined recursively in terms of a parameter-vector  $\vec{m}_w$  of dimension  $w$  having entries  $0, \pm 1$  only.

# Harmonic Polylogarithms

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- The definition of the harmonic polylogarithm is recursive, starting from, for  $\vec{0}_w = (0, \dots, 0)$ ,

$$H(\vec{0}_w; z) = \frac{1}{w!} \ln^w z$$

and for  $\vec{m}_w = (a, m_{w-1}) \neq \vec{0}_w$ ,

$$H(\vec{m}_w; z) = \int_0^z dx f(a; x) H(\vec{m}_{w-1}; x)$$

with

$$f(0; x) = \frac{1}{x}, \quad f(1, x) = \frac{1}{1-x}, \quad f(-1, x) = \frac{1}{1+x}$$

# Multiple Polylogarithms

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- Harmonic polylogarithms are a special case of multiple polylogarithms, which may be expanded as

$$\text{Li}_{k_1, \dots, k_n}(z) = \sum_{m_1 > m_2 > \dots > m_n > 0} \frac{z^{m_1}}{m_1^{k_1} m_2^{k_2} \dots m_n^{k_n}}$$

or expressed as iterated Chen integrals over differential forms  $\omega_0 = dz/z$  and  $\omega_1 = dz/(1-z)$  :

$$\text{Li}_{k_1, \dots, k_n}(z) = \int_0^z \omega_0^{k_1-1} \omega_1 \dots \omega_0^{k_n-1} \omega_1$$

# Harmonic Polynomials

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- The Harmonic polylogarithm may be related to a multiple polylogarithm via the relation

$$\text{Li}_{k_1, \dots, k_n}(z) = \text{H}(\vec{m}; z)$$

with vector  $\vec{m}$  given by

$$\underbrace{(0, 0, \dots, 0, 1)}_{k_1 - 1 \text{ times}}, \underbrace{(0, 0, \dots, 0, 1)}_{k_2 - 1 \text{ times}}, \dots, \underbrace{(0, 0, \dots, 0, 1)}_{k_n - 1 \text{ times}}$$

# Harmonic Sums

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- The proof is based on the fact that the  $\varepsilon$ -expansion of a generalized hypergeometric function expanded about integer parameters may be reduced to a harmonic series of the type

$$\sum_{j=1}^{\infty} \frac{z^j}{j^c} S_{a_1}(j-1) \cdots S_{a_p}(j-1),$$

where  $z$  is an arbitrary argument and the harmonic sums are defined as

$$S_a(j) = \sum_{k=1}^j \frac{1}{k^a}$$

# Generalization

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- Our goal was to attempt to generalize this result to cases where the parameters could be nearly integers or half-integers. In this case, a new type of sum is generated:

$$\sum_{j=1}^{\infty} \frac{1}{\binom{2j}{j}^k} \frac{z^j}{j^c} S_{a_1}(j-1) \cdots S_{a_p}(j-1) S_{b_1}(2j-1) \cdots S_{b_p}(2j-1)$$

These are **generalized harmonic** 0  
**inverse binomial** sums for  $k = 1$   
**binomial** -1

# Generalization

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- ❑ There is presently no proof that a general sum multiple (inverse) binomial sum can be represented in terms of only harmonic polylogarithms.
- ❑ There are, in fact, known examples that cannot be expressed in terms of harmonic polylogarithms with a simple argument.
- ❑ For hypergeometric functions of the **Gauss** type, we have been able to prove a representation in terms of harmonic polylogarithms.

# Gauss Hypergeometric Functions

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- We will discuss a particular class of Gauss Hypergeometric functions, which have the form

$$\begin{aligned} {}_2F_1(A+a\varepsilon, B+b\varepsilon; C+c\varepsilon; z) &= {}_2F_1\left(\begin{matrix} A+a\varepsilon, B+b\varepsilon \\ C+c\varepsilon \end{matrix} \middle| z\right) \\ &= \sum_{j=0}^{\infty} \frac{(A+a\varepsilon)_j (B+b\varepsilon)_j}{(C+c\varepsilon)_j} \frac{z^j}{j!} \end{aligned}$$

where in dimensional regularization,  $d = 4 - 2\varepsilon$ .

[M. Yu. Kalmykov, JHEP04(2006) 256, M.Yu. Kalmykov, B.F.L. Ward, S.A. Yost, JHEP02(2007) 040]

# Gauss Hypergeometric Functions

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- Some Feynman diagrams giving rise to Gauss Hypergeometric Functions include
  - one-loop propagator diagrams with arbitrary masses and momenta
  - two loop bubble diagrams with arbitrary masses
  - one-loop massless vertex diagrams with three nonzero external momenta.
  
- In all these cases, the  $\varepsilon$ -expansions can be written in terms of Nielsen polylogarithms

$$S_{n,p}(z) = \text{Li}_{n+1, \underbrace{1, \dots, 1}_{p-1}}(x)$$

# Theorem

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- We proved the following theorem:

The  $\varepsilon$ -expansion of a Gauss hypergeometric function

$${}_2F_1(A + a\varepsilon, B + b\varepsilon; C + c\varepsilon; z)$$

with  $A, B, C$  integers or half-integers may be expressed in terms of harmonic polylogarithms with polynomial coefficients.

[M. Yu. Kalmykov, JHEP04(2006) 256, M.Yu. Kalmykov, B.F.L. Ward, S.A. Yost, JHEP02(2007) 040]

# Reduction Algorithm

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- The proof begins with the observation that any Gauss hypergeometric function can be written as a linear combination of two others with parameters differing from the original parameters by an integer.
- Specifically,

$$P(a,b,c,z) {}_2F_1(a+I_1, b+I_2; c+I_3; z) = \left\{ Q_1(a,b,c,z) \frac{d}{dz} + Q_2(a,b,c,z) \right\} {}_2F_1(a,b;c;z)$$

with  $a, b, c$  arbitrary parameters,  $I_1, I_2, I_3$  integers, and  $P, Q_1, Q_2$  polynomials in the parameters and argument  $z$ .

# Reduction Algorithm

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- In this way, the given hypergeometric function can be reduced to a combination of five basis functions and their first derivatives:

$${}_2F_1(a, b\varepsilon; 1 + c\varepsilon; z), \quad {}_2F_1(a\varepsilon, b\varepsilon; \frac{1}{2} + c\varepsilon; z), \\ {}_2F_1(\frac{1}{2} + a\varepsilon, b\varepsilon; 1 + c\varepsilon; z), \quad {}_2F_1(\frac{1}{2} + a\varepsilon, b\varepsilon; \frac{1}{2} + c\varepsilon; z), \quad {}_2F_1(\frac{1}{2} + a\varepsilon, \frac{1}{2} + b\varepsilon; \frac{1}{2} + c\varepsilon; z)$$

- In fact, it is known that only the first two are algebraically independent, so to prove the theorem, it is sufficient to consider only these two basis functions and show that they can be expressed as harmonic polylogarithms.

# Outline of Proof

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- The proof proceeds by writing a differential equation satisfied by the basis hypergeometric functions, and expanding the solution in powers of  $\varepsilon^n$ .
- The coefficients of these powers can then be constructed iteratively and recognized as harmonic polylogarithms.
- Obtaining the  $k^{\text{th}}$  coefficient requires knowledge of the previous ones, in this construction.

# Outlook

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- ❑ This is just a very brief introduction to hypergeometric function approach to Feynman diagrams.
- ❑ One goal would be to combine the results into a software package that could be used to simplify Feynman integrals using an algorithm based on this representation.
- ❑ Conversely, mathematicians have been using results motivated by Feynman diagrams to discover new relations among hypergeometric functions and related functions. This is a fertile area of interaction between mathematics and physics.